The qualities of damped responses of RLC circuits

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By Xavier Boluna

ECE 101 :: Professor Sara Abrahamsson :: Juliana Hernandez

Experimentation performed together with Quinn Schmidt

Abstract:

This report explores the behavior of RLC circuits. Parallel RLC circuits have a damping factor inversely proportional to their resistance and capacitance. Conversely, series RLC circuits have a damping factor proportional to their resistance over the inductance.

The level of damping can cause under-, over- and critical damping; each of which have separate and specific behaviors relative to one another.

By creating these circuits and feeding a square wave through the circuit, we can measure the RLC circuits' response and, by varying the resistance, determine which circuits match the qualities of an under-, over- or critically damped system.

This report explores the relationships between their properties and verifies a couples basic mathematical principles of the response on an RLC circuit.

Introduction:

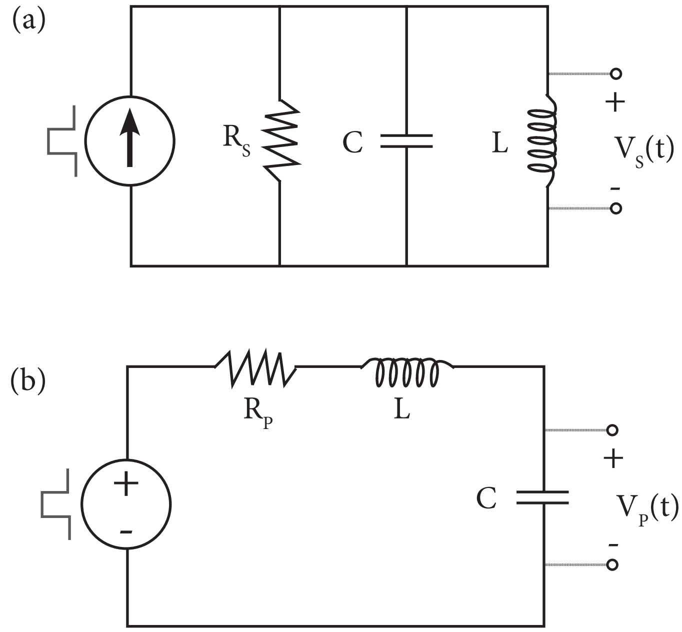
Figure One shows the RLC circuit, which contains either in parallel or in series a resistor (R), inductor (L) and capacitor (C). The subject of this report is the response of an RLC circuit to the square step function of a voltage or current source. This, in essence, simulates the action of flipping a switch on or off, at a certain frequency interval.

Depending on the configuration of an RLC circuit, the response can vary in three primary ways: either by being overdamped, critically damped or underdamped. Each of these cases are dependent on the same fundamental properties of the circuit and exhibit different patterns linked to the rate at which their signals are attenuated. The mathematical basis for their behavior is discussed further in *Methods*. For now, Figure Two suffices to demonstrate the general behavior of overdamped, underdamped and critically damped analog oscillatory signals.

Importantly, a non-damped source has a constant amplitude compared to a damped source, for which amplitude decreases steadily as an exponential function of the damping constant. An underdamped source will continue to oscillate until it converges on equilibrium. A critically damped source does not oscillate, and converges slowly over a single wave. The overdamped source does not equilibrate normally.

Importantly, the damping factor is found from the characteristic equation $s^2 + 2\alpha s + \omega_0^2 = 0$ such that $s = \dfrac{-\alpha\sqrt{\alpha^2 - \omega_0^2}}{2}$.

Figure One: Parallel and series RLC circuits



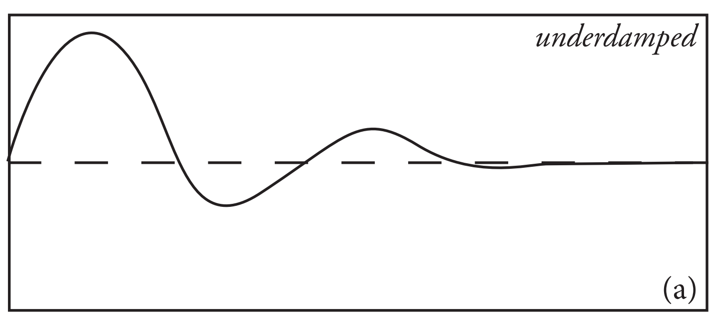
These are the RLC circuits we will be analyzing through this lab report. They are the parallel (a) and series (b) RLC circuits, which each run a voltimeter across a different component: the inductor in the parallel case; the capacitor in the series case.

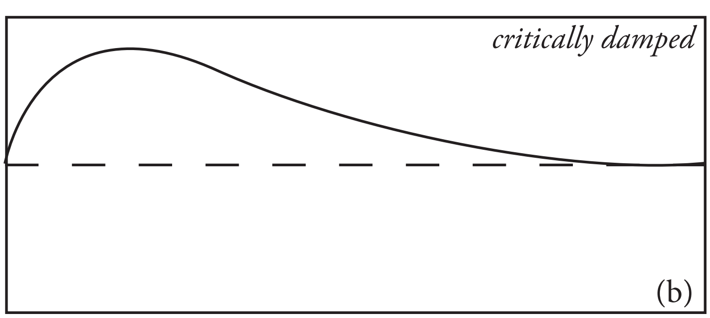
We can define the undamped resonant frequency $\omega_0 = \dfrac{1}{\sqrt{LC}}$ and, looking at Figure One, resolve the energy disappated by the resistor for each the parallel and series cases to be $\alpha_P = \dfrac{1}{2R_P C}$ and $\alpha_s = \dfrac{R_s}{2L}$, respectively.

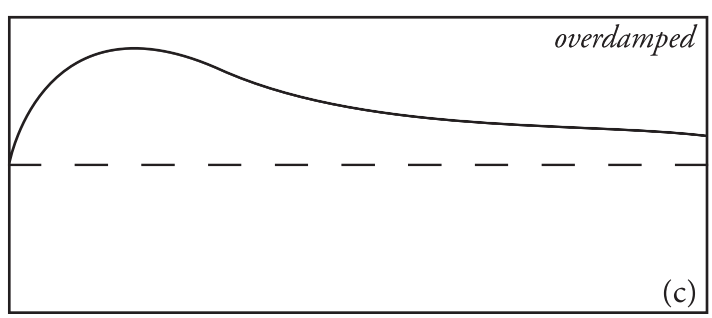
Figure Two: Damped oscillators

There are three types of damped systems: underdamped, critically damped and overdamped. They are qualified by the amount that the wavefunction overshoots or meets the equilibrium point (dashed).

The underdamped case (a) meets and exceeds this point before eventually resting at equilibrium. The critically damped case (b) meets exactly the equilibrium point, with no overshoot. The overdamped case (c) is damped so much that it does not meet the equilibrium point and reaches a different steady-state.







Importantly, we can define the damping qualities of the oscillator in terms of the α coefficient:

$\alpha^2 - \omega_0^2 > 0$ ::*overdamped*

$\alpha^2 - \omega_0^2 = 0$ ::*critically damped*

$\alpha^2 - \omega_0^2 < 0$ ::*underdamped* such that the damped resonant frequency $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 2\pi f_d \sim \frac{1}{\alpha}$ (inversely proportional to the damping factor).

The purpose of this lab report is to assemble each of the schematics in Figure One and, by varying the resistor, apply each different damping case to verify and investigate the damping behavior of RLC circuits.

We determine our resistance by means of solving for a given value ζ where $\zeta = \frac{\alpha}{\omega} = 0.5, 1, 2$. A value of 1 will represent the critically damped case, whereas the under- or overdamped cases will be represented when $\zeta \neq 1$.

In order to comprehensively analyze the damping qualities of these waveforms, we will need to gather a few parameters. Each of them will have a rising-time, or time within which 10 and 90% of the equilibrium voltage is reached. We can use this data to compute the dead-time $\tau = \frac{t_1 - t_2}{\ln(\frac{v_1 - v_f}{v_2 - v_f})}$ (or indeed any arbitrary locale t_1,t_2 for which we can measure respective voltages).

We will also note overshoot, or the percentage by which the maximum exceeds the steady-state.

Figure Three gives an example of the underdamped case for a square wave. Underdamped cases also have a discernible period, which we can measure as well.

Methods:

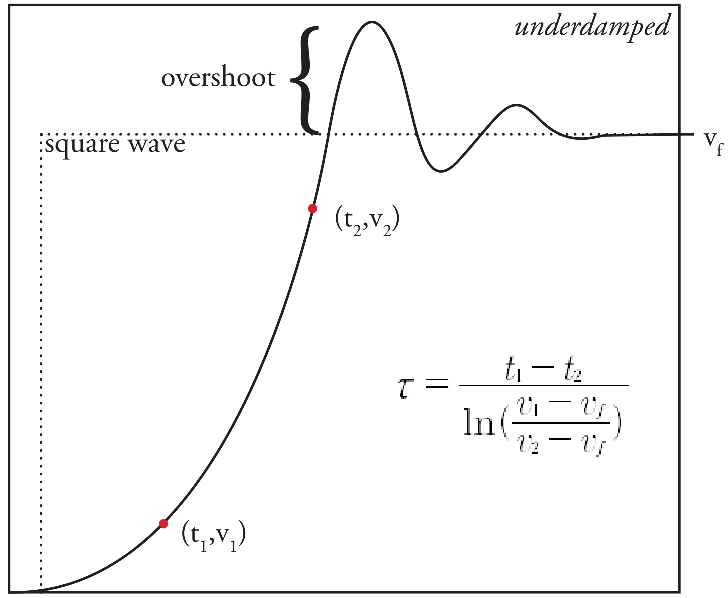
Using all of the underlying mathematics from above, we need only define our values for the capacitor and inductor. From there we can calculate our undamped resonant frequency which informs our choice of frequency for our square wave generator.

We chose a capacitor rated 1 nF and an inductor rated 10 μH. Therefore, our undamped resonant frequency ω_0 is roughly 0.3 megaHertz/2π. We can therefore choose a frequency of about a factor slower to allow time to for the oscillator's behavior to be fully displayed -- about 1 kHz should suffice.

From here, we need to combine the equations for α and ζ to solve for the resistances needed to observe certain behavior for each circuit. Solving algebraically and solving, we get the following correspondent resistances for the parallel circuit R_p = 100, 50 and 25Ω. Similarly for the series circuit, R_s = 100, 200 and 400Ω.

There are two additional principles we will consider when analyzing these circuits. The first is to verify that $\alpha \sim f_d$. As α increases, we can expect fewer cycles to

Figure Three: The underdamped case - example

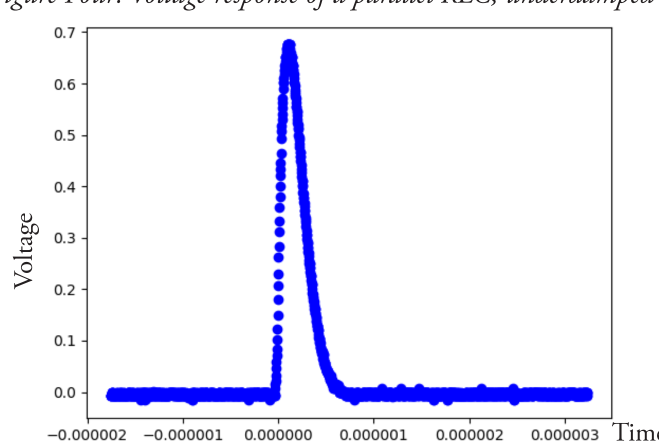


The overdamped case, illustrating the analysis of any individual damped response. We measure overshoot and dead-time of each individual waveform; period in addition for every underdamped case.

reach the steady-state. second, we will experimentally determine R_C, representing the resistance necessary to produce critical damping, for each the parallel and series circuits.

Specifically in the case of the parallel circuit, the response will occur more familiar form when represented as the current across the inductor. As shown in Figure One (a), the signal generator itself is a current. The voltage reading across the inductor, such as the critically damped case, appears like that shown in Figure Four. We need to convert our voltage reading to a current waveform, and we can use the following relationship to determine the current value i_n at each individual value of time n such that $i_n = \frac{1}{L}\sum_0^n v_n$.

Figure Four: Voltage response of a parallel RLC, underdamped



As we can see, the voltage steady-state over the inductor is at zero. As such, the spike makes evaluation of our parameters more difficult.

Two last notes worth mentioning on the methodological side of things involve the signal generator and the data collection methods. The first is that the signal generator itself contributes roughly 50 Ω of resistance. For this reason, when we represent R_P or R_S, they already include the contribution of the signal generator; we simply create the circuit representing the signal generator as a parallel or series resistor. Especially in the parallel case, for example, we can represent a signal generator in parallel with another 50 Ω resistor as an equivalent 25 Ω resistor.

The second is that, when collecting data for the parallel case, it becomes necessary to analyze the data using programming tools. For this reason, analysis for the parallel circuit is performed using Python while the analysis of the series circuit is done in-situ with the oscilloscope.

Results:

Let's start with the parallel case.

At 25 Ω, we get the underdamped case as as shown in Figure Five. Let's break down the involved analysis. We can easily measure the period of oscillation for this underdamped case at the first and second peak, for which we resolve the period $T = 6.45 \times 10^{-7}$ s = 645 ns.

Next is the choice of an equilibrium position. This is easiest to do with the oscilloscope, by measuring the equilibrium position of the entire square wave, and applying that information to our graph.

Next, we take the maximum and minimum of the graph. The difference between the max and equilibrium will represent our final current i_F. Lastly, we take two points in roughly 10% and 90% zones of our signal and calculate voltage differences from the minimum for each and the time difference between them. This gives us all the parameters necessary for the calculation of τ, which we find to be $\tau = 9.93 \times 10^{-8}$ s = 99.3 ns for this case.

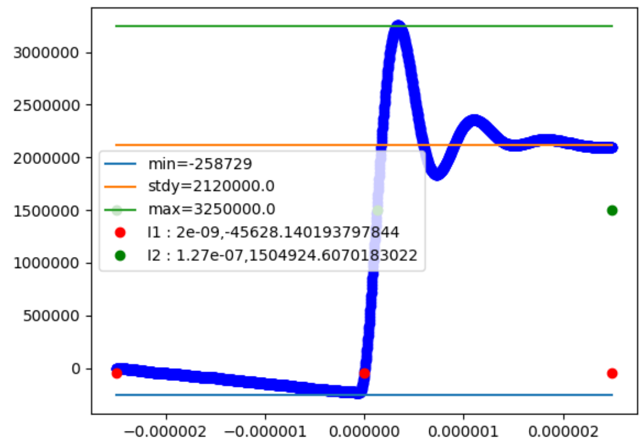
We also take the ratio of the maximum over the equilibrium voltage to find our overshoot at 34%.

This analysis for dead-time and overshoot are applicable to the remaining two cases identically.

As such, we move onto the 50 Ω, critically damped case shown in Figure Six. As we can expect, the overshoot is visibly quite smaller and there is no additional oscillation.

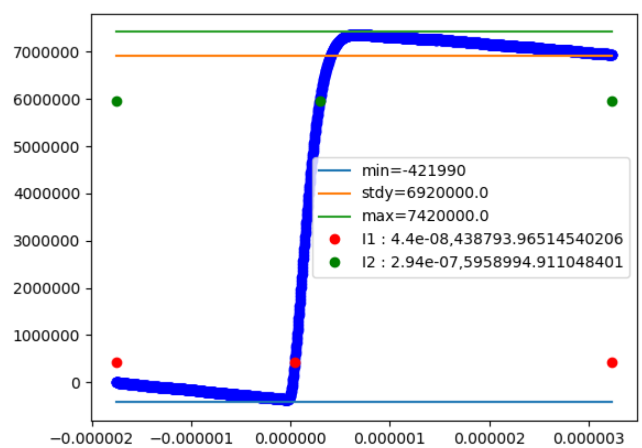
Using identical methodology to the underdamped case, we find the overshoot to be just 6.7% and the dead-time $\tau = 1.31 \times 10^{-7}$ s = 131 ns.

Figure Five: Underdamped response of parallel RLC circuit



The underdamped current response of the inductor in a parallel RLC circuit, labeled with the relevant information needed to analyze for overshoot and dead-time. In these cases, we calculate maximum, final and relevant voltages as a difference from the minimum voltage of the waveform.

Figure Six: Critically damped response of parallel RLC circuit



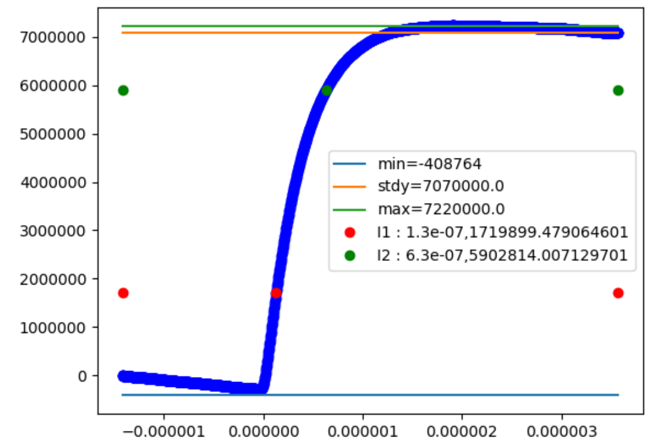
The critically-damped current response of the inductor in a parallel RLC circuit, labeled with the relevant information needed to analyze for overshoot and dead-time. The rigorous damping effect minimizes the overshoot and prevents any additional oscillation outside of the main overshoot.

As the critically damped case does not have any additional oscillations, a calculation for period is infeasible. It is worth discussing, however, how much this choice of R_C abides by the criterion for critical damping. We approached the critical damping point by use of two different methods: the calculation mentioned earlier in *Methods*, and the use of a potentiometer to observe the continuous change in the damping factor. Approaching the critical 'appearance,' we were able to verify that the appropriate 50 Ω theoretical resistance is accurate to our expectations of how critical damping should appear. Specifically, the small overshoot and gradual but accelerating approach of the steady-state are significant indicators.

At 200 Ω, we get the overdamped case, shown in Figure Seven. This case has a noticeably smaller overshoot, with a more gradual peak and a much slower tendency towards the steady-state. In fact, the equilibrium current listed for the overdamped case does not necessarily represent the steady-state. The signal continues to damp slowly throughout the length of the packet.

Using identical analysis as before, we find the relative overshoot to be just 2%. The waveform's dead-time is $\tau = 3.28 \times 10^{-8}$ s = 328 ns

Figure Seven: Overdamped response of parallel RLC circuit



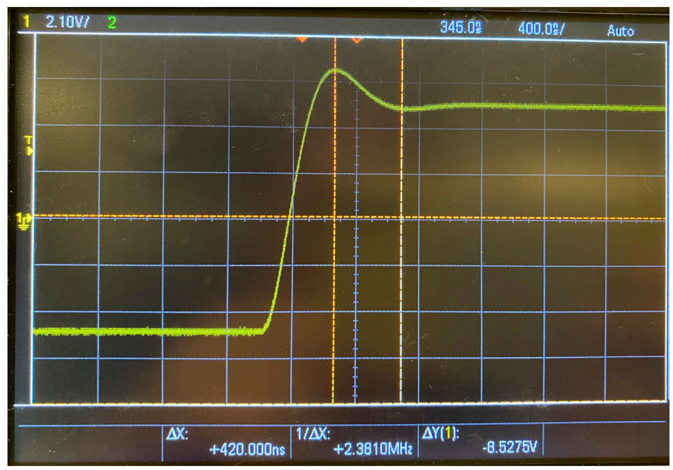
The overdamped current response of the inductor in a parallel RLC circuit, labeled with the relevant information needed to analyze for overshoot and dead-time.

Looking towards the series circuit, Figure Eight shows the underdamped case with a resistance R_S of 100 Ω. It's important to note that, here, we deal with voltage rather than current. The oscilloscope is placed across the capacitor terminals as opposed to the inductor. As we can see clearly, the period of oscillation is 840 ns.

Taking an identical approach to that performed in the parallel cases, we obtain an overshoot of 15% and a dead-time τ = 119 ns.

The instantaneous reading of the oscilloscope allows for us to install a potentiometer and quite easily see the variation of the amplitude and period of the oscillating component of the underdamped waveform. With this, we observed that the smaller the resistance, and therefore the smaller the resistor energy loss $\alpha \sim R_S$, the higher the damped angular frequency ω_d. As we know $f_d \sim \omega_d$, this verifies that α is proportional to the inverse of f_d.

Figure Eight: Underdamped response of series RLC circuit

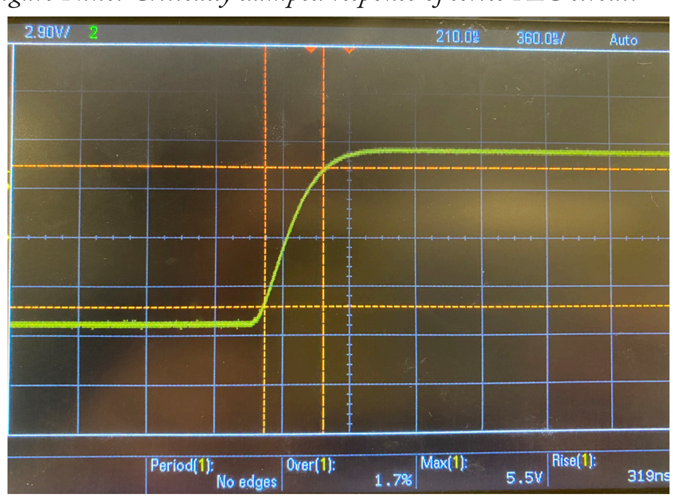


An image of the oscilloscope reading for the underdamped response of a series RLC circuit. This particular image shows the cursors placed around the first peak and trough of the oscillation. Doubling this, we can calculate the period.

For the critically damped case, we choose a resistance 200 Ω, which we experimentally verify using the same methodology as described in the critically damped parallel circuit.

As shown in Figure Nine, we get an overshoot of just under 2% and a dead-time of roughly 162 ns.

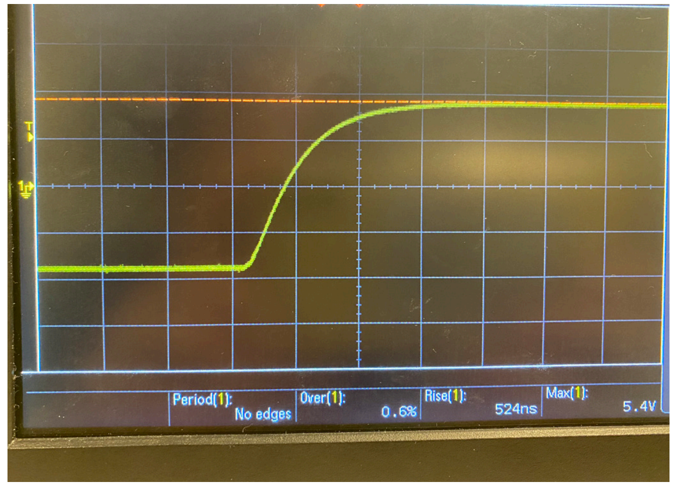
Figure Nine: Critically damped response of series RLC circuit



This critically damped case demonstrates a much smaller overshoot and a no discernible oscillations as the waveform peaks. Instead, it slowly decays back to the steady state.

Our last case is the overdamped series RLC circuit at 400 Ω, which is shown in Figure Ten on the next page. As we can see, there is a very minimally discernible overshoot, which is calculated to be just 0.7%. The dead-time comes out to 471 ns.

Figure Ten: Overdamped response of series RLC circuit



The underdamped case is very distinguishable. There is practically no overshoot and the curve is very shallow. The maximum is slightly higher than the steady-state, and approaches very slowly.

Discussion of Results:

The results of this experiment can be measured by two metrics: first, the analysis of the individual cases with respect to α, f_d and f_0; second, with respect to our additional principles that $\alpha \sim 1/f_d$ and the verification of R_C for our circuits.

With each of our analyses, we were able to verify the relationship α has with $\omega_0 \sim f_0$ as described by the underlying mathematics. By discovering under-, critically and over-damped cases by calculation of α and ζ, we verify the conceptual basis. Additionally, we strengthen the case for the second principle, which is experimentally confirmed in the critically-damped parallel circuit.

Having a theoretical basis for α, we can explore its relationship in detail with f_d. As explained by the underdamped case for the series circuit, we can verify the inverse proportionality between the them. We can, however, also logically think through the circuits' form as a result of the damping factor. As the value of α increases from a low value (underdamped), the oscillations decrease until they are practically absent (critically-damped) and, finally, fully non-present (overdamped). Both the parallel and series circuits demonstrate this relationship.

As such, while this isn't a deeply quantitative analysis of the phenomena described in this paper, the somewhat qualitative conclusions made from the analysis of these responses meets our standards for the verification of our principles and the investigation of the behavior of these waveform responses.

Conclusion:

This report's goal was to explore the responses of parallel and series RLC circuits to a square wave as they experience under-, critically- and over-damped behavior. We connected these behaviors to their correspondent mathematical relationships to the energy lost by the resistor and the undamped resonant frequency of the circuit. In doing so, we can evaluate the degree to which a signal will be damped and choose resistances to target these behaviors in experimental settings.

We were able to analyze the qualities of each of these cases and, importantly, were able to verify the mathematical relationships between them. The success of this experiment, as mentioned before, isn't in the quantitative results but rather the verification of these mathematical principles and the core concepts behind them.