The Cavendish Experiment

An reproduction of the experiment to derive a universal constant of gravitation

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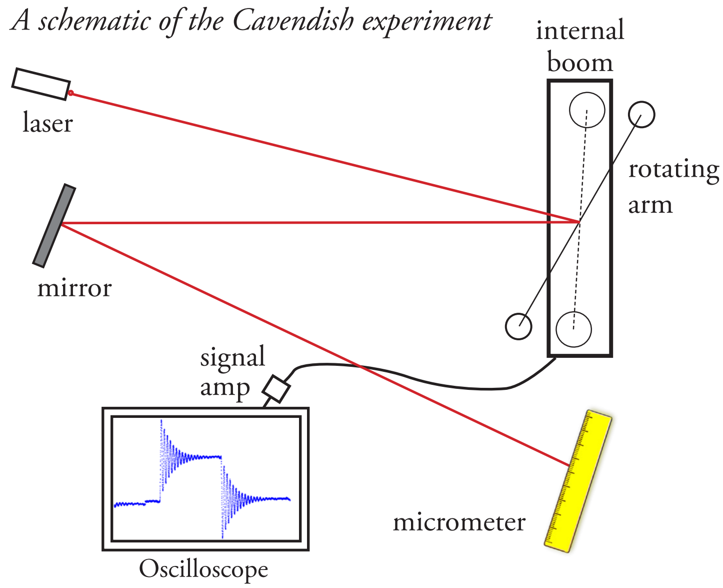
ABSTRACT

This report recreates the Cavendish experiment, performed first by Henry Cavendish in 1797. The goal of the experiment is to measure the density of the Earth, which mathematically corresponds to the universal constant of gravitation. The apparatus used to do so leverages the mutual attraction of lead balls to oscillate a suspended boom by which the periodicity and angle of equilibrium allow for the derivation of the gravitational constant.

This report reproduces the experiment and, using modern electronics, deduce the gravitational constant. We resolved a value of $G = (6.859 \pm 9.9) \times 10^{-11}\ m^3kg^{-1}s^{-2}$, which corresponds to a one-tailed p-value of 49.26% indicating significant agreement with the reference value provided by the NIST (see Appendix).

This report offers comparisons also to Cavendish's measurements of the density of the Earth and describes in detail the mathematical and methodological principles required to understand and reproduce the experiment.

A schematic of the Cavendish experiment



Introduction

In 1797, Henry Cavendish set out on a quest that would have seemed impossible just a century before. Even today it still seems fantastical to say the Cavendish experiment attempeted to *weigh the world*.

Newton, about 110 years before, published his laws on gravitation *Philosophiæ Naturalis Principia Mathematica*, in which he proved the inverse square law of gravitation $F_g \sim 1/r^2$. He took a stab at measuring the relative masses of cosmic bodies among the solar system, but his attempts were only approximates.

Cavendish's own experiment was the first to accurately measure the density of Earth -- or in his time, the relative density with reference to water (Cavendish, 1797). This was the first experiment to directly measure the gravitational attraction of two masses and it was from this bedrock that the concept of the gravitational constant came to light nearly another century later.

Cavendish's apparatus consisted of two pairs of lead balls; one larger set suspended statically with another smaller pair suspended so they could twist about the center. The balls' attraction to one another caused the smaller balls to twist until the string's torsion counterbalanced the gravitational attraction. Knowing the torsion quality of the string, and the separation of the balls from one another, Cavendish could effectively derive some version of the gravitational constant, and thereby the density of the Earth.

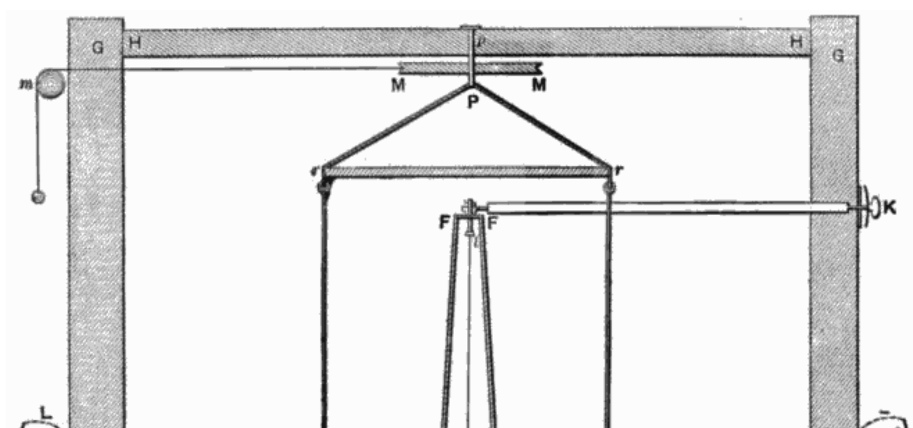
Cavendish's final value was 5.448 times that of water's density, which we know today as 5.448 g/cm^3. Using the reverse of the equation derived in Theorem One, we calculate Cavendish's gravitational constant to be 6.74×10^{-11} m^{3}kg^{-1}s^{-2}. The accepted value we will use is 6.67430×10^{-11} m^{3}kg^{-1}s^{-2}; just 1% fractional difference, illustrating the accuracy of Cavendish's measurement.

The beautiful implication of this result is that the Earth's total density measures roughly 80% less than his value. This was the first suggestion that the core of the Earth was composed of a different material than its mantle. With the help of seismology, scientists would learn that the Earth's protective magnetic field is powered by a dynamic shell of molten iron at its core (Feynman, 1963).

In this paper, we repeat Cavendish's experiment, using modernized methods and technology. We use a fundamentally similar apparatus to calculate the gravitational constant of Earth and thereby the density of Earth. We compare our derived value to that of accepted science today, and offer recommendations on how this experiment can be most accurately reproduced. Lastly, we explore the context and utility of the gravitational constant in Modern Physics to demonstrate its ubiquity and the impact of its discovery on history.

This report is organized in the following sections, including data tables and bibliographical information placed in the Appendix:

Cavendish's Apparatus



Taken from Cavendish's *Experiments to determine the Density of the Earth*, this schematic describes the design of his apparatus.

The larger pair of lead balls, labeled W, are suspended statically. The smaller lead balls are then placed an angle θ away are allowed to twist their central wire until the force of torsion from the wire counterbalances their gravitational attraction.

From this, knowing the balls' respective masses, the wire's torsion coefficient and the balls resting separation from one another, it is possible to derive the gravitational constant.

Methods and Procedures

The star child of the Cavendish experiment is the boom. Just like the original apparatus built in 1797, our main box suspends on a wire a pair of lead balls in balance such that any perturbations, gravitational or otherwise, induce a swivel about the center. A mirror, placed at the end of an opening in the box, allows the boom to reflect a laser shined through it. As the boom rotates, this boom will trace a path with its deflections which we can measure.

Another static arm is placed outside the box, tipped with a pair of lead balls. This is the arm which we move to induce gravitational attraction.

After moving this arm from one extreme to the other, the mutual attraction from the lead balls induces rotational oscillations in the internal boom arm. These oscillations, as in (b), deflect the laser onto a micrometer, with which we can visually see and measure the positional change over time. This allows us to measure the angle the boom makes with its normal by use of the methods outlined in Figure Two (next page). Over time, the boom arm equilibrates as the torsional force of the tungsten wire equals the gravitational attraction of the balls, and the system becomes static.

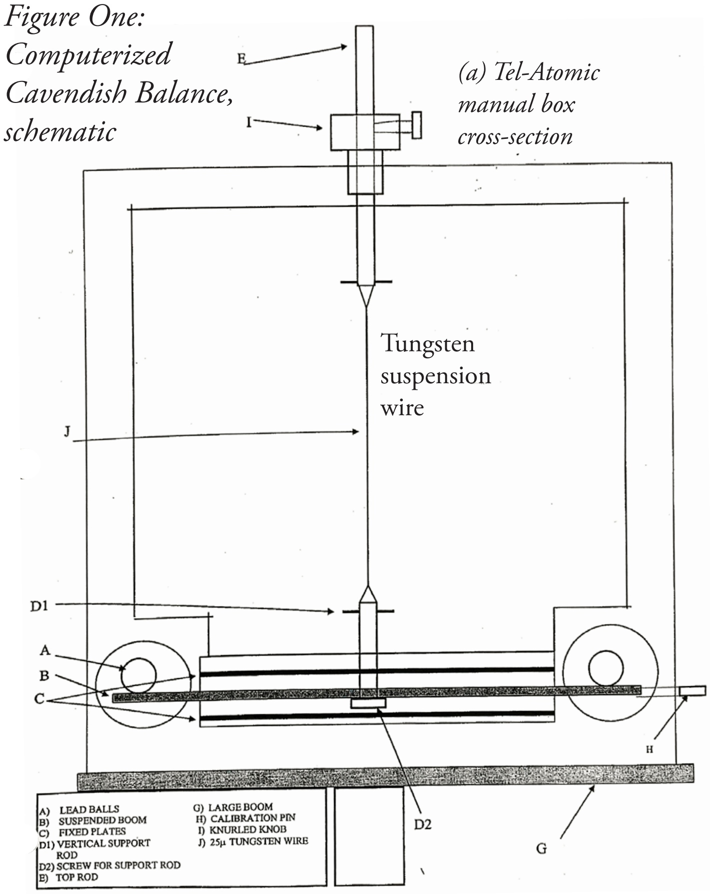
Our modern twist on the affair is the integration of an electrical component to read these oscillations as a change in potential over time. This allows us to record continuous, high-resolution data and trace out the damped waveform as it occurs in the box, as seen in (c).

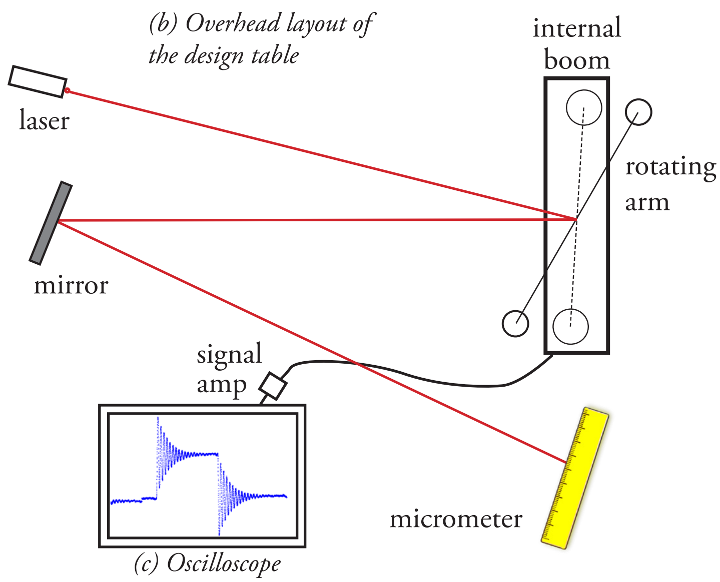
In order to know how to perform our measurements, we first need to know what information is needed to obtain the gravitational constant. The workings to do so are described in detail in the subsequent section, located on the next page. The takeaways are the following:

T	Period, obtained from the oscillation data
L	The path length of the laser
x	The deviation between equilibrium positions on the micrometer
R	The distance between the lead balls' centers
d	The length of the boom arm from the rotation axis to the internal lead ball
M	The mass of the external lead ball

The latter three values are characteristic of our apparatus and are provided to us by the Tel-Atomic manual. They are housed within the fragile apparatus and are thereby difficult to independently verify. We assume that these values are minute enough that their contributions are outweighed by other error (see *Error Analysis*).

Figure One: Computerized Cavendish Balance, schematic





(a) A diagram of the Tel-Atomic *Cavendish Apparatus* provided for this experiment. This is a cross-section similat to Figure One in which the two lead balls are shown, balanced perfectly across a tungesten suspension wire. The oscillating boom (such that one or the other lead ball arm comes out of the page) twists that wire.

(b) A full overhead view of the experiment reveals many constituent components, including a laser which the internal boom deflects, changing the reading on the micrometer. The overall distance travelled by the laser is lengthened by the mirror so that the reading is more precise. The oscilloscope reads the electrical signal generated by the moving boom and allows us to take precise records.

Period (T) is measured by analyzing the oscillation data obtained with our oscilloscope. The software methodology will be elaborated upon further in the results section.

The path length of the laser (L) was traced out with a thread and then subsequently measured; the error contributed is the result of the 'stretchiness' of the string.

Lastly, the change in position across the micrometer (x) can be measured as one would. Error is determined by comparing the deviation across the multiple measurements obtained for the same position.

Our methodology is therefore rather simple. Starting from one position in equilibrium, we record the micrometer position. This requires great precision and, as such, we took great lengths to record multiple different datapoints for each equilibrium position.

Hitting record on our oscilloscope data, we can then move the rotating arm to the opposite position in equilibrium. The oscilloscope will register sweeping paths which will be mirrored in the behavior of the laser on the micrometer. Eventually, the boom reaches equilibrium, and we record micrometer data and repeat the process as we return the boom arm to the the initial position. The oscilloscope recording all the while, we allow this second change to equilibrate and record the micrometer data there, too.

Derivation: Boom equilibrium angle and G

At first glance, measuring the gravitational constant on the order of 10^{-11} using masses in the order of half a kilogram seems miraculous. In fact, the relationship between them is such that we can derive it explicitly with few assumptions.

First, it is necessary to describe Newton's Law of Gravitation:

$$F_g = G\frac{m_1 m_2}{r^2}$$

where the value r represents, in our case, the distance from each center of a lead ball to the other. Appendix A describes the motivation behind the center of a sphere being equivalent to a point mass of the same magnitude as the Shell Theorem, proven originally by Newton in *Principia*.

The torque on a boom is described by the equation:

$$\tau = d \times F$$

wherein d describes the distance of the lead ball to the pivot arm and F describes whatever external force motivates the torque: in this case, F_g. We must remember, however, that there are two lead balls on each end of the boom, such that:

$$\tau = d(2F_g) = 2d(G\frac{m_1 m_2}{r^2})$$

Over time, our boom equilibrates and rests at a deflection angle θ_d from the 'normal' position (if no other masses were present). The torsion constant of the tungsten wire *K* is thereby defined as:

$$K = \frac{\tau}{\theta_d} = \frac{2d}{\theta_d}G\frac{m_1 m_2}{r^2}$$

It's useful now to define each mass as the large and small lead balls to which we give *M* and *m* respectively. We can solve for G using algebra:

$$G = \frac{K}{2d}\frac{r^2}{Mm}\theta_d$$

For each equilibrium position, the boom advances θ_d twice. Additionally, for each θ_d, the boom deflects an angle $2\theta_d$. Figure Two describes this relationship such that the total distance traced by the laser from one equilibrium position to the next, *x*, relates: $\theta_d = \frac{x}{4L}$

Now, *K* itself is related to the moment of inertia by the undamped resonant frequency, which is itself motivated by the damped resonant frequency:

$$\omega_\phi^2 = \frac{K}{L} = \omega_{observed}^2 + b^2$$

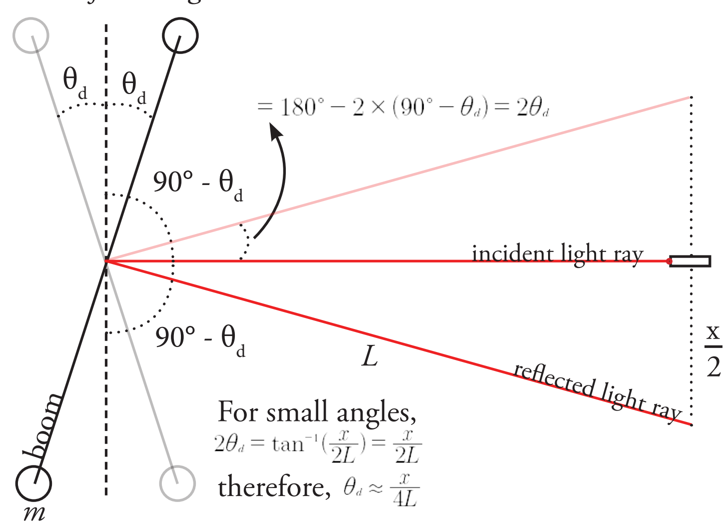
where b is the damping constant.

As such, we find the moment of intertia I which, for a small lead ball, involves:

$$I = \sum(I_i + A_i d_i^2) = 2m(d^2 + \frac{2}{5}r^2) + \frac{m_b}{12}(l_b^2 + w_b^2)$$

where for each segment (boom arm and lead ball) *i* we get its moment of inertia, area and distance from the pivot arm. In our case, the ball of radius r and the aluminum arm of length l_b and mass m_b.

Figure Two: The relationship between internal boom angle and deflected light



This simplified schematic (excluding the mirror) outlines the relationship by which the angle θ_d advances for the deflected laser is $2\theta_d$. As such, considering laser travel *x*, we determine $\theta_d \approx \frac{x}{4L}$.

As the mass of the beam m_b = 0.0085 kg and is divided by 12, and that r = 0.00672 m and is squared, we can make the presumption that these terms contribute relatively little to the moment of intertia. This allows us to simplify *I* to be:

$$I \approx 2md^2$$

We know that the damped oscillation frequency will correspond with our observed period and so, plugging in *I*, we can deduce:

$$K = I(\omega_{observed}^2 + b^2) \approx I[(\frac{2\pi}{T_{observed}})^2]$$

$$K \approx 2md^2(\frac{2\pi}{T})^2$$

Which allows us to define a value for the torsional constant of the tungsten wire. Plugging this into the overall equation for the gravitational constant, we get:

$$G = \frac{K}{2d}\frac{r^2}{Mm}\theta_d$$

$$G \approx (\frac{2\pi}{T})^2\frac{r^2 d}{M}\theta_d$$

wherein our calculations for the value of G will reference this equation.

A keen eye will notice that, because of our assumptions, the small lead mass *m* disappears. While technically under a more stringent derivation, this value would still be involved, the assumptions made when reducing the moment of intertia are perfectly sound and, therefore, this derivation is sufficently accurate.

One last derivation to make is that of the density of Earth in relationship with the gravitational constant. We obtain accepted values for the local gravitational acceleration *g* and the radius of the Earth R_{Earth} (NSSDC, 2019) and treat them as constants.

$$g = G\frac{M_{Earth}}{R_{Earth}^2}$$

$$M_{Earth} = g\frac{R_{Earth}^2}{G}$$

$$\rho_{Earth} = \frac{M_{Earth}}{V_{Earth}} = \frac{M_{Earth}}{\frac{4\pi}{3}R_{Earth}^3} = \frac{3g}{4\pi G R_{Earth}}$$

Practical considerations

Methodologically, taking data for the Cavendish experiment is rather straightforward. There are, however, logistical elements that need attention to ensure that the measurement is accurately taken.

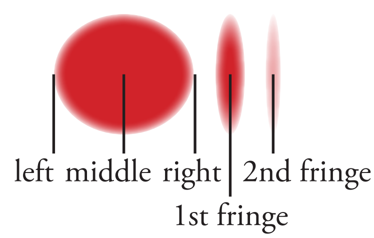
The first involves minimizing noise. With an instrument as sensitive as the Cavendish experiment, the potential for noise to distrupt measurements cannot be underrated. Our apparatus was sensitive to every minute bump, and sometimes even heavy footfall. In fact, from six complete datasets, only the last made it into this report because, for one reason or another, noise affected the data.

As such, it is beneficial not only to take extreme care, but to also be recording the oscilloscope data long before the experiment is performed. We often ran about 10 minutes of static data before running the experiment. This is beneficial because it gives you an understanding of the magnitude of the ambient noise for that dataset and gives ample time for any additional factors to present themselves.

Corollary to this point, provide ample time between changes in the external rotating arm's position for equilibrium to be fully reached. Rotating the arm before this equilibrium is reached will contribute to the noise of the subsequent measurement.

Another crucial consideration is the micrometer measurements. In our apparatus, the laser travelled a length of nearly 5 meters. Over this distance, the width of the laser beam increases significantly -- so much so, that any measurement that gauges the 'middle' could deviate by more than half a millimeter, or up to 10%. For this reason, we chose multiple reference points to provide some average of the values with their statistical deviation representing the error in the average. The features we pegged our value to were the left, middle and center of the beam along with first- and second-order fringes produced by the light's interference with itself.

Figure Three: Measurement locations of the laser beam



The length of the path that the laser travels causes the beam to widen significantly. In order to account for its width, and therefore the variation in measurements it can cause, we take multiple measurements from different locations to resolve an average value and statistical deviation.

Error Analysis

Some discussion must specifically be had over the choice of error, its propagation, and our standards for determining whether our measurement is in good agreement with theory.

Firstly, with data we record ourselves, we obtain its variance σ^2. Such is the case with our micrometer measurements *x* and our determination of the period T. For each of these, we take an average of the values we collect as our best value. We determine the error to be the square root of the set's variance, or its standard deviation. Say some value q has a variance σ_q^2 -- its error will be $\sigma_q = \delta q$. We can represent this value as $q_i \pm \delta q_i$.

The data which we obtain from the Tel-Atomic manual is rated by the manufacturer of the apparatus. As mentioned earlier, the apparatus is sufficiently fragile that disassembling it simply to verify these values would be difficult (and likely unappreciated by our laboratory supervisors). As such, we take their given constants at face value. Adding uncertainty to the precision of the value proves mathematically inconsequential when we propagate error. For this reason, we choose simply to take the values as they are. The error we produce in our other measurements adequately drowns out these values' contributions.

When propagating error, we will use the rule of quadrature. For some function $q(x_1,x_2,...,x_n)$ with associated errors δx_i, the error for that function is:

$$\delta q = \sqrt{\sum_i^n \left(\frac{dq}{dx_i}\delta x_i\right)^2} = \sqrt{\left(\frac{dq}{dx_1}\delta x_1\right)^2 + ... + \left(\frac{dq}{dx_n}\delta x_n\right)^2}$$

For each equation, we repeat this process so that we arrive on a final value for the gravitational constant, with associated error $\delta G_{experimental}$.

We source our 'accepted' scientific value for the gravitational constant from Comittee on Data for Science and Technology's internationally recommended values, which are backed at the very least by the United States' scientific community (NIST, 2018). Their recommended value $G_{reference} = (6.67430 \pm 0.00015) \times 10^{-11}$ $m^3kg^{-1}s^{-2}$.

The weight σ for our Gaussian Distribution will then be the error propagated from our G_{exp} and G_{ref} by the same methods mentioned above.

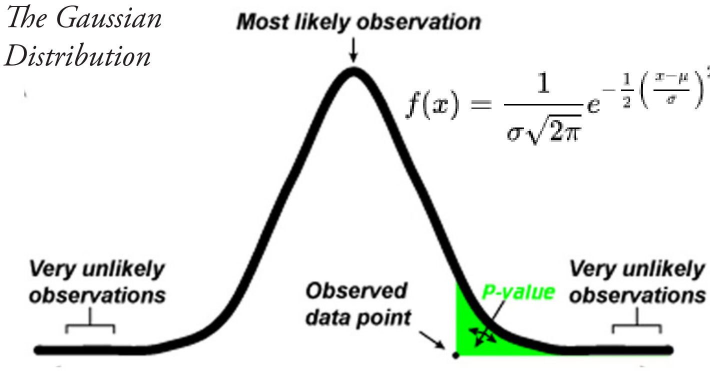
Looking at Figure Eight, we see the equation for the Gaussian distribution, which defines its own error parameter σ and a center for the function, μ. We want to center our distribution at $x = 0$ so we choose $\mu = 0$.

The coefficient of the Gaussian Distribution normalizes the distribution, such that the integral over its full bounds (-∞,∞) resolves to one. For this reason, the p-value suggests a percent likelihood that a given value is correctly obtained.

Results

Having gone into depth with our methodology, the derivation of our equations and the context by which we will evaluate our results, we turn towards the collection of data and analysis of the results.

Figure Four: The Gaussian Distribution



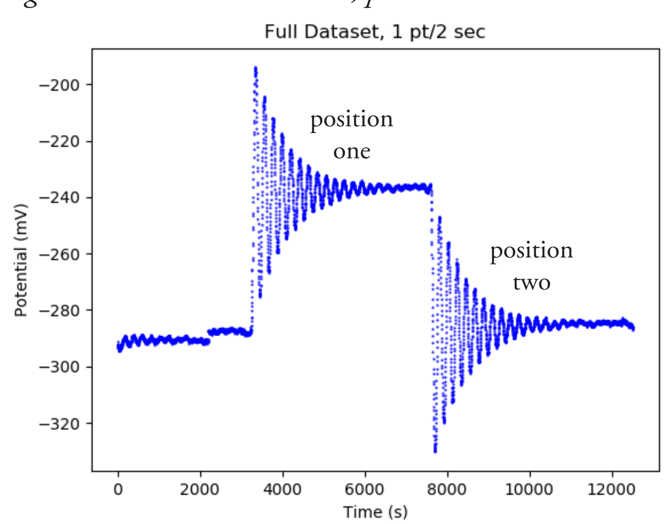
The Gaussian Distribution takes in our combined uncertainty to produce a distribution which evaluates the probabilistically 'best' value. With this, we can calculate the area under the curve from our data point (which will be the deviation from the accepted value) and find a p-value. This value informs our evaluation of the accuracy of experimental results.

Finding the period of oscillation

Having exported our dataset from the oscilloscope, we can plot it fully, as in Figure Five. The value in seeing this set fully laid is the immediate sense of the pattern and the damping factor. As the system continues to settle down, its period remains roughly the same while its amplitude and axis of oscillations converge on a single potential.

The oscilloscope converts the boom data to an electrical potential which is amplified by the signal amplifier seen in Figure One (b). As such, the most relevant information we can glean has purely to do with time.

Figure Five: The Raw Dataset, plotted



Taking the data recorded from the oscilloscope, we can plot the full waveform produced by our boom through time. We have independently verified the exact times which we moved the rotating arm with the spikes in oscillation.

We can extrapolate from this dataset the period of oscillation rather easily. Using a quick corner-searching methods, we can highlight all of the points at which there appears to be a local extremum.

Splitting these into two different wave packets, we get the first (upper) waveform in Figure Six (a) and the second (lower) waveform in (b).

As is noticeable, our algorithm for searching the extremum is imperfect, and picks up many more extrema as noise overtakes the waveform in later stages. As such, we identify by hand the discernible first 10 period measurements for both the minima (dashed line) and maxima (dotted) of the waveform.

We can compute an average period and statistical deviation for each of these sets for both waveforms. As the period of oscillation should be dependent on the moment of inertia and therefore the physical apparatus, we can assume that the period will be the same for every trial. For this reason, we average all the sets and find their standard deviation to be T = 213.9 ± 5.2 seconds.

Finding the angle of deflection

The second crucial component we need is the deflection angle θ_d. This, as we discussed earlier, requires the length of the laser *L* and the path traced by the laser on the micrometer *x*.

Having measured *L* with a string, we can measure it piecewise with a meter stick and, accounting for the minimum and maximum stretchiness of the string, get the values 3,623 and 3,619 millimeters. We can average these and consider the boundaries as the error, such that *L* = 3,621 ± 2 millimeters.

We can then take the measurements of *x* from equilibrium to position one to equilibrium in position two. This table references the positions in Figure Three.

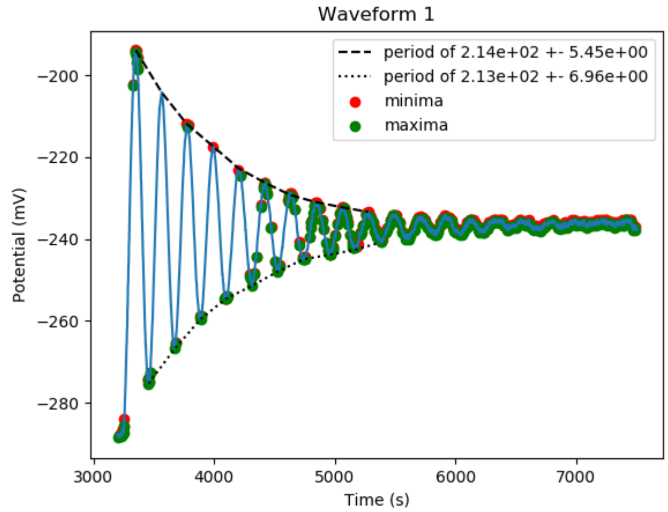
Location	Equilibrium position one	Equilibrium position two	Total travel (mm)
Left	3.986	-3.991	7.977
Middle	2.433	-5.461	7.984
Right	1.181	-6.715	7.896
1st fringe	0.646	-7.063	7.709
2nd fringe	-0.026	-7.896	7.87

Taking the average value and error of the total travel, we get *x* = 7.869 ± 0.088 millimeters.

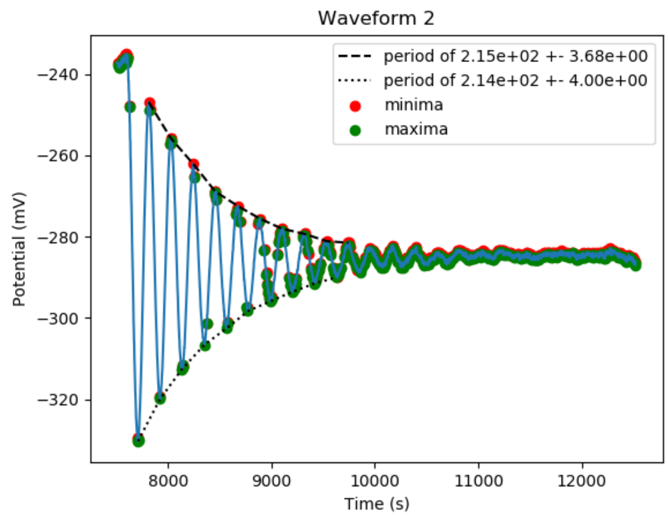
We use these values to solve the equation for θ_d and propagate error such that $\theta_d = (5.433 \pm 0.786) \times 10^{-4}$. This is the most sensitive part of the analysis; while the error seems high, it is not entirely unexpected.

Figure Six: Periodicity analysis on both wave packets

(a) Analysis of first waveform



(b) Analysis of second waveform



Analyzing these waveforms' maxima and minima is assisted by an algorithm that looks for corners in the graph. The extrema are then handpicked -- ten points per side; twenty per graph -- in order to calculate the average difference and the standard deviation between them.

The minima are traced with the upper dashed line while the maxima are traced with the dotted line.

Determining the torsional constant

Our next step is to obtain the torsional constant of our tungsten wire, defined earlier as:

$$K \approx 2md^2\left(\frac{2\pi}{T}\right)^2$$

The Tel-Atomic manual gives us a mass for the small balls *m* = 14.7 grams and its distance from the rotation axis d = 66.56 millimeters.

Given our obtained values and propagating for error, we obtain a torsional constant $K = (112.4 \pm 0.87) \times 10^{-10}$ m^2 kg s^{-2}.w

Determining the universal gravitational constant

Our last step is to calculate, finally, the gravitational constant of the universe. We utilize the equation:

$$G \approx (\frac{2\pi}{T})^2 \frac{r^2 d}{M} \theta_d$$

The Tel-Atomic manual gives us the mass of the large lead ball M = 917 grams.

Using this value, and those we have obtained before, we resolve $G_{exp} = (6.859 \pm 9.9) \times 10^{-11}\ m^3 kg^{-1} s^{-2}$.

We now need to create a Gaussian distribution for this experimental value with respect to the reference value we obtained earlier as $G_{reference} = (6.67430 \pm 0.00015) \times 10^{-11}\ m^3 kg^{-1} s^{-2}$.

Propagating their combined error into a weight for the distribution and centering it about zero, we get Figure Seven.

As shown by the red vertical lines, our experimental value's deviation from the reference is minimal compared to the width of the distribution.

We take the integral between the positive deviation and infinity to resolve a one-tailed value p = 49.26%.

This is extremely good, suggesting our measurement is probabilistically very near the best possible value. This adds support to our theory and methodology.

Discussion

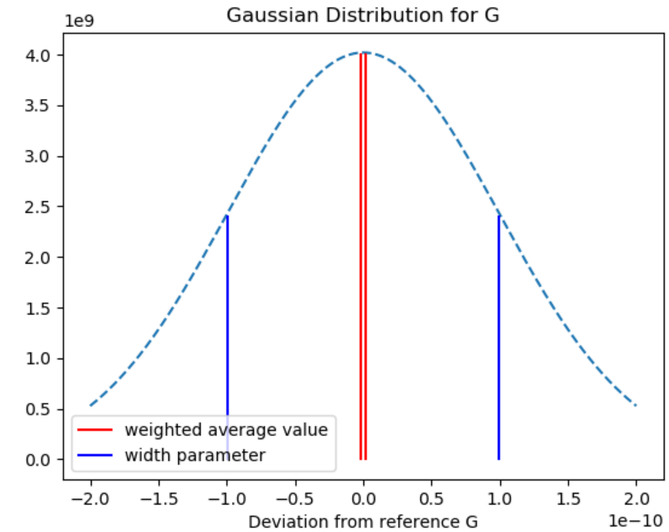
The goal of this venture was to recreate Cavendish's experiment by the use of modernized means. By this standard, I would consider this experiment a success.

We derived the equations equivalent to what Cavendish would have worked with through his discovery nearly two centuries ago. Using an apparatus fundamentally similar to his, but with electronic elements which allows us to plot with precision the oscillation of the boom, we calculated the gravitational constant within good agreement with the current accepted scientific value and theory.

It is important to note, however, that the agreement of this value is dependent on the error associated with it. For our comparison, the error in our experimental value drowned out the error in the reference value, meaning that the high level of agreement reflects moreso the likelihood that a measurement with our error could have been accurately measured.

Cavendish, in 1797, specifically calculated the specific density -- or density relative to water. Keeping in mind his original value of 5.448 g/cm^3, we calculate the density of Earth using our new experimental value

Figure Seven: Gaussian distribution to determine accuracy of $G_{experimental}$ with respect to $G_{reference}$



This distribution describes the deviation from the reference value for our experimental value of the gravitational constant. The Gaussian Distribution is a normalized distribution which consider a null hypothesis for our value and attempts to evaluate its probabilistic agreement.

for the gravitational constant. Our experimental density resolves to be $\rho_{experimental}$ = 5.354 g/cm^3.

While in Cavendish's time, the gravitational constant had not yet been synthesized in the form we recognize it as, his scientific work represented a big step in the history of understanding gravity and the Earth. Further, more accurate measurements of the gravitational constant have relied on methods founded by Cavendish's apparatus and leverage identical principles.

Future reproductions of this experiment could benefit from the stringent methodology described in the Methods section. The main contributors of error in our experiment were the length of the laser path L and the oscillation period T. The former could benefit from a less stretchy string so that the measurement varies less. The latter would require less noise and more data points to narrow down the variation.

It's worth noting that Cavendish's experiment was performed in a sealed room with a larger apparatus and larger masses than ours (Encylopaedia Britannica). Though he didnt have the benefit of modern technology, he did have more resources and more space. Our lab was ventilated, populated with others including ourselves and was situated on a metal (vibrationally conductive) table. As such, our measurement environment was conducive to some noise, noise which was demonstrated in the data as multiple peaks in an extremum or a difficulty in resolving further extrema as the total amplitude of the wave decreased (which is why we only took 20 measurements of extrema per wave).

The implications of Cavendish's discovery are grand. First, of course, was the discrepancy between the density of the Earth's mantle and its overall density. This revelation belied the Earth's liquid iron core. This discovery ,and further seismological developments to discover the solid 'cool' core within it, linked the Earth's magnetic field to the its dynamic core.

Mars has no magnetic field and is therefore unable to retain an atmostphere or protect its surface from solar radiation. The perspective we glean from Earth's dynamic core points to the possibility that Mars has a less-liquified core and, in its past, had perhaps a more dynamic core, an intrinsic magnetic field and a healthier atmosphere (NASA, 2016). This increases the chance that life once existed in Mars' pasts, and heralds a warning for the challenges Earth could face should its core cool to a certain point in the distant future.

Cavendish's experiment reaches further than Earth science itself, with the gravitational constant being ubiquitous in our search for understanding of the natural forces of the universe.

A worthwhile exercise which appreciates the sensitivity of the experiment is to calculate the ratio of the electrostatic force to the gravitational force. We can take the case of two bare protons separated by a distance r. Knowing their masses and charge are the same, we can consider:

$$\frac{F_e}{F_g} = \frac{k\frac{2q}{r^2}}{G\frac{2m}{r^2}} = \frac{kq}{Gm}$$

Plugging in with the real values of proton charge and mass, the Coulombic constant and our gravitational constant, we get a ratio of 1.26×10^{28}. The electrostatic force is demonstrably more powerful than the gravitational force.

In essence, though the particulars of the Cavendish experiment could be considered mundane, in fact it is the perspective through which we see nature which is fundamentally important. The theory of gravitation and the search for a gravitational constant ask questions that are deeply intrinsic to human experience. What causes objects to fall and celestial objects to move in such patterns? What lies beneath the Earth under my feet? How does the universe create itself? What are its rules; its book of instructions?

Perhaps most humbling of all is that the very same gravitation we measured between two balls less than a few centimeters wide governs the entire, churning cosmos, from nebulae to neutron stars.

Appendix

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The Shell Theorem

Proven in Newton's Principia in the early 16th century, the Shell Theorem proves that, in terms of gravitational attraction, a spherical object of uniform density can be considered as a point mass at the center of that sphere.

Though Newton proved this theorem using explicit geometry, in fact the easiest way to arrive at this theorem is by using Gauss' Law.

Gauss' Law for gravity describes:

$$\oint g \cdot dS = -4\pi G m$$

where M is a uniformly dense mass inside a sphere of radius r and g is its gravitational field.

We can describe this closed integral as the dot product of a the gravitational acceleration as a factor of distance r and the normal unit vector:

$$\oint g \cdot dS = \int g(r) \cdot \hat{n}\, dS$$

Take note that as the dot product with the normal unit vector requires exclusively the outward-pointing vectors, and g(r) is dependent exclusively on the distance r from the center of the sphere, we can consider g(r) to be normal and independent of dS. As such, we can solve the integral of the closed sphere which resolves the area of the sphere, $4\pi R^2$:

$$g(r) \int dS = g(r) 4\pi r^2$$

Then, taking the original proposition of Gauss' Law and equating it to our solved integral:

$$g(r) 4\pi r^2 = -4\pi G m$$

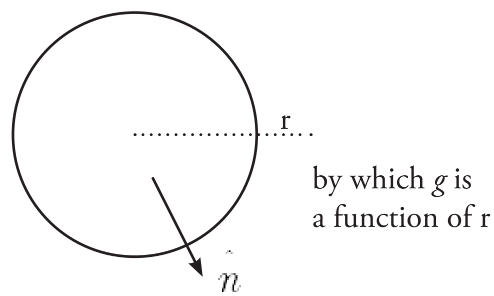
Solving for g(r), we get:

$$g(r) = -\frac{Gm}{r^2}$$

This is the law of gravitation for a single body, i.e. the gravitational acceleration towards a mass *m*. Just as F=ma, we can use the gravitational acceleration to solve the force of gravitational attraction between any two masses.

This proves that the theorem is true for any spherical surface in which a uniformly dense mass is enclosed. As such, the Shell Theorem is proven.

Cross-section of a sphere of uniform mass density *m*



This cross-section of a sphere visualizes the normal vector and the radius r. The function g(r) describes the gravitational acceleration as a function of radius and acts exclusively as a magnitude normal to the sphere's surface.