1 Problem 1

1D particle in potential $V(\hat{x}) = Usin(2\pi\hat{x}/a)$ for $U > 0$. Particle is on ring of length $L = Na$ for N integer, and satisfies periodic boundary condition $\Psi(x) = \Psi(x + L)$.
a. $< x|p >= \frac{e^{ipx/\hbar}}{\sqrt{L}}$.

$$\begin{aligned} e^{\frac{i}{\hbar}(xp)} &= e^{\frac{i}{\hbar}p(x+L)} \\ e^{ixp} &= e^{ixp+iLp} \\ e^{iLp} = 1 \to p &= \frac{2\pi}{L}n \text{ for n integer.} \end{aligned} \tag{1}$$

b. Need to find $< p|V(\hat{x})|p' >$. Re-express $V(\hat{x})$ as

$$\begin{gathered} V(\hat{x}) = U[\frac{e^{2\pi i\hat{x}/a} - e^{2\pi i\hat{x}/a}}{2i}] \text{ and plug in as} \\ < p|V(\hat{x})|p >= \frac{U}{2i}[< p|e^{2\pi i\hat{x}/a}|p > - < p|e^{2\pi i\hat{x}/a}|p >] \\ \text{For each, } < p|e^{\pm 2\pi i\hat{x}/a}|p' >= \int_0^L \frac{e^{-ipx}}{\sqrt{L}} e^{\pm 2\pi i\hat{x}/a} \frac{e^{ip'x}}{\sqrt{L}} dx = \frac{1}{L}\int_0^L e^{-ix(p-p'\mp 2\pi/a)} dx \end{gathered} \tag{2}$$

Digressing to the possible solutions to this integral:

$$\begin{gathered} \text{If } p - p' = \pm\frac{2\pi}{a} \to \frac{1}{L}\int_0^L e^0 = L/L = 1 \\ \text{else } p - p' \neq \pm\frac{2\pi}{a} \to \frac{1}{L}[\frac{e^{-ix(p-p'\mp\frac{2\pi}{1})}}{-i(p - p' \mp \frac{2\pi}{a})}]|_0^L = \frac{i}{L}[\frac{e^{iL(-p+p'\pm\frac{2\pi}{a})} - 1}{(p - p' \mp \frac{2\pi}{a})}] \\ \text{wherein from a. } p = \frac{2\pi n}{L} = \frac{2\pi n}{Na} \text{ therefore } \frac{i}{L}[\frac{e^{i2\pi(-n+n'\pm N)} - 1}{i\frac{2\pi n}{LNa}}] = 0 \\ \text{Recalling that } e^{i2\pi N} = 1 \\ \text{As such,} \begin{cases} p - p' = \pm\frac{2\pi}{a} \to 1 \\ p - p' \neq \pm\frac{2\pi}{a} \to 0 \end{cases} \end{gathered} \tag{3}$$

We enforce the first case $p - p' = \pm\frac{2\pi}{a}$ such that

$$< p|V(\hat{x})|p' >= \frac{U}{2i}[< p|e^{2\pi i\hat{x}/a}|p > - < p|e^{2\pi i\hat{x}/a}|p >] = \frac{U}{2i}[\delta_{p-p',\frac{2\pi}{a}} - \delta_{p-p',\frac{-2\pi}{a}}] \tag{4}$$

c. Derive $|\Psi >$ and reduce it to $|p = 0 >$ when $U \to 0$ to first order in U. By definition provided in lecture, the wavefcn is approximated to O(V) as below

$$
\begin{aligned}
|\Psi >&\approx |0> + \frac{-Q}{E_0 - H^0} V(\hat{x})|\Psi^0> \\
&= |0> + \frac{-\sum_{p'!=0} |p'><p'|V|0>}{E_0 - \frac{p'^2}{2m}}
\end{aligned} \tag{5}
$$

From last, substitute $\sum_{p'} |p'><p'|V|0> = \frac{U}{2i}[|\frac{2\pi\hbar}{a}> - | - \frac{2\pi\hbar}{a}>]$ so that $E_0 = 0$ while $H^0 = \frac{p^2}{2m} = \frac{(2\pi\hbar/a)^2}{2m} = \frac{\hbar^2}{2ma^2}$.

$$
\begin{aligned}
|\Psi >&\approx |0> + \frac{1}{\frac{p'^2}{2m}} \frac{U}{2i} [|\frac{2\pi\hbar}{a}> - | - \frac{2\pi\hbar}{a}>] \\
&= |0> + \frac{Uma^2 i}{\hbar^2} [|\frac{2\pi\hbar}{a}> - | - \frac{2\pi\hbar}{a}>]
\end{aligned} \tag{6}
$$

Which we can verify reduces to $|0>$ as $U \to 0$.

d. To verify as a block wave, we need to express as plane wave × periodic function f(x) = f(x+a).
Reexpress the results of 1c as below

$$
\begin{aligned}
<x|\Psi> = <x|0> + C <x|\frac{2\pi\hbar}{a}> - C <x|\frac{-2\pi\hbar}{a}> &\approx e^0 + Cie^{2\pi i x} - Cie^{-2\pi i x} \\
&= 1 + 2iC\sin(2\pi x)
\end{aligned} \tag{7}
$$

Which is periodic due to the sine function and has a plane wave block attached where e^{ikx} as $k \to 0$.

e. $p = \pm\frac{\hbar\pi}{a}$ therefore the subspace is $[\frac{-\hbar\pi}{a}, \frac{\hbar\pi}{a}]$ with elements $< \pm\frac{\hbar\pi}{a}|V| \pm \frac{\hbar\pi}{a}>$.

$$
V = \begin{bmatrix} <\frac{\hbar\pi}{a}|V|\frac{\hbar\pi}{a}> & <-\frac{\hbar\pi}{a}|V|\frac{\hbar\pi}{a}> \\ <\frac{\hbar\pi}{a}|V|-\frac{\hbar\pi}{a}> & <-\frac{\hbar\pi}{a}|V|-\frac{\hbar\pi}{a}> \end{bmatrix} = \frac{U}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{8}
$$

Therefore eigenvalues are $\pm\frac{U}{2}$ and the eigenvector is $\begin{bmatrix} 1 \\ \pm i \end{bmatrix}$. The resulting wave equation takes the form $\Psi> = |\frac{\hbar\pi}{a}> \pm i| - \frac{\hbar\pi}{a}>$ which, normalized, becomes

$$
|\Psi_\pm> = \frac{1}{\sqrt{2}}\left[|\frac{\hbar\pi}{a}> \pm i|\frac{-\hbar\pi}{a}>\right] \text{ with eigenenergies } \frac{p^2}{2m} \pm \frac{U}{2} = \frac{\hbar^2\pi^2}{2ma^2} \pm \frac{U}{2} \tag{9}
$$

2 Problem 2

a. We want $\vec{\boldsymbol{A}} = 0$ therefore we impose $\vec{E} = -\partial_z \phi \hat{\boldsymbol{z}} = E\hat{\boldsymbol{z}}$ where we choose E in $\hat{\boldsymbol{z}}$. Therefore the scalar potential is $\phi = -Ez$ and the Stark correction is

$$\delta H_S = -e\phi = eEz \tag{10}$$

wrt uniform electric field $\vec{E}$ and elementary charge e.

$$H = H_0 + \delta H_S = \frac{p^2}{2m} - \frac{e^2}{r} + eEz \tag{11}$$

b. Firstly, we know $E_0 = -13.6eV$. The first-order energy is $E_1 =< \phi_{100}|H_S|\phi_{100} >= eE < 100|z|100 >$ where the expectation of of an odd fcn is 0, therefore $E_1 = 0$. The second-order energy is $E_2 = \sum_{nlm\neq 100} \frac{|<\phi_{nlm}|H_S|\phi_{100}>|}{E_0 - \frac{-13.6eV}{n^2}}$. Therefore,

$$E \approx -13.6eV + \sum_{nlm\neq 100} \frac{e^2E^2n^2}{-13.6eV(1-n^2)} | < \phi_{nlm}|z|\phi_{100} > |^2 \tag{12}$$

c. Firstly, we can say that $\hat{R}(\pi)$ behaves functionally like $\hat{P}$ in that $\hat{R}(\pi)\hat{R}(\pi) = I = \hat{P}\hat{P}$ where I is identity. As such, we can confirm our earlier conclusion for $E_1 = 0$ with an alternative proof that

$$\begin{gathered} < nlm|z|100 >=< nlm|\hat{P}\hat{P}z\hat{P}\hat{P}|100 > \\ = (< nlm|\hat{P})(\hat{P}z\hat{P})(\hat{P}|100 >) \\ = ((-1)^l < nlm|)(-z)((-1)^0|100 >) = (-1)^{l+1} < nlm|z|100 > \\ (-1)^{l+1} = 1 \therefore \text{l is odd to be nonzero} \\ < 100|z|100 >= 0 \end{gathered} \tag{13}$$

As such, the first-order energy dies and the second-order energy is restricted to $nlm \neq 100$, l odd.
Next, we recognize $\hat{R}^{-1}\hat{R} = I$ and choose a rotation $\psi = 2\pi\mathbb{I}$ by which $R(\psi) = e^{-i\hat{L}_z\psi}$.

As such,

$$
\begin{gathered}
< nlm|z|100 >= (< nlm|\hat{R}^{-1})(\hat{R}z\hat{R}^{-1})(\hat{R}|100 >) \\
= (< nlm|e^{i\hat{L}_z\psi})(z)(e^{-i\hat{L}_z\psi}|100 >) \\
\text{Via the theorem described in class, } \Psi_{nlm}(\hat{R}r) = e^{im\psi}\Psi_{nlm}(r) \\
\text{satisfies } e^{iL_z\psi}|nlm >= e^{im\psi}|nlm > \\
\therefore < nlm|z|100 >= (< nlm|e^{i\hat{L}_z\psi})(z)(e^{-i\hat{L}_z\psi}|100 >) =< nlm|e^{im\psi}|z|e^0|100 > \\
< nlm|z|100 >= e^{im\psi} < nlm|z|100 >
\end{gathered} \tag{14}
$$

Wherein with $e^{im(2\pi\mathbb{I})} = 1$, the only way which this statement is true is if m is zero. The second-order energy becomes a sum over $nlm \neq 100$, l odd and m=0.

d. $H_z = \frac{|e|}{m_e c}\vec{S}.\vec{B}$ for n=1 is $|nlm >\rightarrow |100 > \otimes| \uparrow>$ and $|100 > \otimes| \downarrow>$. We choose $\vec{B} = B.\hat{z}$ such that $\vec{S} \rightarrow \hat{S}_z$ so that each $| \uparrow>, | \downarrow>$ eigenvector has eigenvalue $+\frac{h}{2}, -\frac{h}{2}$ for $\hat{S}_z$ such that the eigenenergies are $\frac{|e|B\hbar}{2m_e c}, \frac{|e|B\hbar}{2m_e c}$ and we add the ground state order -13.6eV to each.