1 Problem 1

$$x, y, z \in [-\frac{L}{2}, \frac{L}{2}], \text{ for } p = \hbar k$$
$$H^0 = \frac{p_x^2}{2m_x} + \frac{p_y^2}{2m_y} + \frac{p_z^2}{2m_z} \text{ with } <r|p> \frac{e^{ik\cdot r}}{\sqrt{Volume}} \tag{1}$$

a. Density of states $\tilde{\rho}(E,\theta,\phi)dEd\Omega = \frac{d\tilde{N}(E,\theta,\phi)}{\text{Volume}}$ for $d\Omega = \sin\theta d\theta d\phi$.

$$Q = \sum_{f!=i} = |V_{fi}|^2 \delta(E_f - E_i)$$
$$\text{Taking this to the limit of an integral } dEd\theta d\phi : \tag{2}$$
$$Q = \text{Volume} \cdot \int dE_f d\Omega \cdot \tilde{\rho}(E,\Omega) \left[<\vec{p_f}|V_{fi}|\vec{p_i}>\right]^2 \delta(E_f - E_i)$$

b. Usual DOS is $\frac{dN(E)}{\text{Volume}} = \rho dE$. The modified case $\frac{d\tilde{N}(E)}{\text{Volume}} = \tilde{\rho} dE d\Omega$. Therefore,

$$\int \tilde{N} d\Omega = N \ \rightarrow \ \int d\Omega \cdot \tilde{\rho}(E,\Omega,\phi) = \rho(E) \tag{3}$$

c. With the isotropic case ($m_x = m_y = m_z = m$), we take $p_x = p_y = p_z = p = \sqrt{2mE}$:

$$\rho = \int d\Omega \tilde{\rho}(E) = \int \tilde{\rho}(E) sin\theta d\theta\phi = 4\pi\tilde{\rho}$$
$$\tilde{\rho} = \frac{1}{4\pi}\frac{d}{dE}\frac{4\pi p^3}{3(2\pi\hbar)^3} = \frac{d}{dE}\frac{(\sqrt{2mE})^3}{3(2\pi\hbar)^3} \tag{4}$$
$$= \frac{m^{\frac{3}{2}}\sqrt{E}}{2^{\frac{5}{2}}(\pi\hbar)^3}$$

d. The following assumes $m_x = m_y$ for the density of states ($\propto$ E $\propto$ m) to be manipulated as symmetric over Ω. By definition in class, $\sum |V_{ni}|^2 = \frac{\int d\Omega |V_{ni}|^2}{4\pi} = \{|V_{ni}|^2\}_{E_f}$. Recall $\rho = 4\pi\tilde{\rho}$.

$$Q = \text{Volume} \cdot \int dE_f \cdot \tilde{\rho}(E_f) \cdot 4\pi \cdot \int d\Omega \frac{1}{4\pi} \left[<\vec{p_f}|V_{fi}|\vec{p_i}>\right]^2 \delta(E_f - E_i)$$
$$= \text{Volume} \cdot \int dE_f \cdot \rho(E_f) \cdot \{|V_{ni}|^2\}_{E_f} \cdot \delta(E_f - E_i) \text{ QED} \tag{5}$$

e. Consider $\{|V_{ni}|^2\}_{E_f}\delta(E_f - E_i)$ the average over final states of momentum p_n –

The momentum is composed of $\vec{p} = (p_x, p_y, p_z)$, but for each direction (x,y,z), $E_x = p_x^2/2m$, etc. That means both the initial and final states require:

$$E_x = \sqrt{2m_x p_x}, E_y = \sqrt{2m_y p_y}, E_z = \sqrt{2m_z p_z} \tag{6}$$

As such, the momentum-space is a 3d ellipsoid and the initial state and final states will be at different points on the ellipsoid: $(r_i, \phi_i, \theta_i) \rightarrow (r_f, \phi_f, \theta_f)$. Obviously if the system is assumed to be isotropic, we eliminate the radial component and the states lie on a sphere.

f. Make use of $< r|p >= \frac{1}{\sqrt{\text{Vol}}} e^{i\vec{k}\cdot\vec{r}}$ and $\vec{p} = \hbar\vec{k}$.

$$< p_n|V|p_i >= \int_{\text{Vol}} \frac{e^{-i\vec{k_n}\vec{r}}}{\sqrt{\text{Vol}}}(Ue^{-|\hat{r}|/a})\frac{e^{i\vec{k_i}\vec{r}}}{\sqrt{\text{Vol}}} d^3\vec{r} = \frac{U}{\text{Vol}} \int_{\text{Vol}} e^{-i\vec{r}(\vec{k_n}-\vec{k_i})} e^{-|\hat{r}|/a} d^3\vec{r}$$

$$\text{With } L >> a \text{, allow} \int_{\text{Vol}} d^3\vec{r} \rightarrow \int_0^\infty dr \int_0^{2\pi} d\phi \int_{-1}^1 d\text{cos}\theta.$$

$$\begin{aligned}
\text{Pick } \vec{k_n} - \vec{k_i} \text{ in } \hat{z}\text{: } &= \frac{U}{\text{Vol}} \int_0^{2\pi} d\phi \int_0^\infty \int_{-1}^1 e^{-i|k_n-k_i|r\text{cos}\theta} d\text{cos}\theta r^2 e^{-r/a} dr \\
&= \frac{2\pi U}{\text{Vol}} \int_0^\infty r^2 e^{-r/a} \left[\frac{e^{-i|k_f-k_i|r} - e^{i|k_f-k_i|r}}{-i|k_f-k_i|r}\right] dr \\
&= \frac{2\pi U i}{|k_f-k_i|\text{Vol}} \int_0^\infty r \left[e^{-r(1/a+i|k_f-k_i|)} - e^{-r(1/a-i|k_f-k_i|)}\right] dr \\
&= \frac{2\pi U i}{|k_f-k_i|\text{Vol}} \left[\frac{-4i|k_f-k_i|}{a(|k_f-k_i|^2+(1/a)^2)^2}\right] = \frac{8\pi U}{\text{Vol}} \left[\frac{1/a}{(|k_f-k_i|^2+(1/a)^2)^2}\right]
\end{aligned} \tag{7}$$

g. Compute $\{|V_{ni}|^2\}_{E_f}$ We align k_i in $\hat{z}$ and re-express $\vec{k} = k \begin{bmatrix} \text{sin}\theta\text{cos}\phi \\ \text{sin}\theta\text{sin}\phi \\ \text{cos}\theta \end{bmatrix}$.

That means $|k_f - k_i|^2 = k^2[(1-\cos^2\theta) + \sin^2\theta] = 2k^2(1-\text{cos}\theta)$.
Sub in for the formula of

$$\begin{aligned}
< p_f|V|p_i > &= \frac{8\pi U}{\text{Vol}} \left[\frac{1/a}{(2k^2(1-\text{cos}\theta)+(1/a)^2)^2}\right] \\
&= \frac{8\pi U}{\text{Vol}} \left[\frac{1/a}{(2k^2+(1/a)^2)^2}\right] \left[\frac{1}{1-\frac{2k^2}{(1/a)^2+2k^2}\text{cos}\theta}\right]^2 \\
&\text{Where we use } \alpha = \frac{2k^2}{r_0^{-2}+2k^2} < 1 \\
< p_f|V|p_i > &= \frac{8\pi U}{\text{Vol}} \alpha^2 \frac{1}{r_0 4k^4} \frac{1}{(1-\alpha\text{cos}\theta)^2}
\end{aligned} \tag{8}$$

As such,

$$\begin{aligned}\{|<p_f|V|p_i>|^2\}_{E_f} &= \frac{1}{4\pi}\int\left[\frac{8\pi U}{\text{Vol}}\alpha^2\frac{1}{4r_0k^4}\frac{1}{(1-\alpha\cos\theta)^2}\right]^2 d\Omega \\ &= \frac{1}{4\pi}\left[\frac{2\pi U\alpha^2}{\text{Vol}}\frac{1}{r_0k^4}\right]^2\int_{-1}^{1}d\cos\theta\int_0^{2\pi}d\phi\frac{1}{(1-\alpha\cos\theta)^4} \\ &= \frac{1}{4\pi}\left[\frac{2\pi U\alpha^2}{\text{Vol}}\frac{1}{r_0k^4}\right]^2 2\pi\frac{1}{3\alpha}\left[\frac{1}{(1-\alpha)^3}-\frac{1}{(1+\alpha)^3}\right] \\ &= \frac{1}{6\alpha}\left[\frac{2\pi U\alpha^2}{\text{Vol}}\frac{1}{r_0k^4}\right]^2 6\alpha\frac{1+\alpha^2/3}{(1+\alpha^2)^3} \\ &= 4\left[\frac{U\pi\alpha^2}{\text{Vol}r_0k^4}\right]^2\frac{1+\alpha^2/3}{(1+\alpha^2)^3}\end{aligned} \tag{9}$$

h. The units of a quantity are referred to as [qty].

- $[\alpha]$ = unitless therefore $[\alpha^2\frac{1+\alpha^2/3}{(1+\alpha^2)^3}]$ = unitless
- $[r_0]$ = [a] = length
- Volume is obviously length^3
- $k = \frac{p}{\hbar}$ therefore [k] = $\frac{\text{(mass)(length)}}{s}\frac{1}{\text{energy}\cdot s} = \frac{1}{\text{length}}$
- As such, $[\frac{1}{\text{Vol}k^4r_0}] = \frac{\text{length}^4}{\text{length}^3\cdot\text{length}}$ = unitless
- Therefore, the units depend on U^2 in which [U] = energy.

This agrees with the fact that V_{fi} should have units of energy such that $[|V_{fi}|^2] = [U^2]$ = energy^2.

2 Problem 2

a. In a box Vol = L^3, derive $\int_{\text{Vol}}\vec{B}^2(\vec{r},t))d\vec{r} = \sum_{\vec{k}\alpha}\vec{k}^2\left[c_{\vec{k}\alpha}c_{-\vec{k}\alpha} + c_{\vec{k}\alpha}\overline{c_{\vec{k}\alpha}} + \text{c.c.}\right]$ using $\vec{B} = \nabla\times\vec{A}$ with $\vec{A}(\vec{r},t) = \frac{1}{\sqrt{\text{Vol}}}\sum_{\vec{k}\alpha}\left[c_{\vec{k}\alpha}(t)\vec{u}_{\vec{k}\alpha}(\vec{r}) + \text{c.c.}\right]$.

Make use of $\vec{u}_{\vec{k}\alpha}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\cdot\epsilon^{(\alpha)}{}_{\vec{k}} = \overline{\vec{u}_{-\vec{k}\alpha}(\vec{r})}$ and note that $c_{\vec{k}\alpha}(t) = c_{\vec{k}\alpha}(0)e^{i\omega_{\vec{k}}t} \not\propto e^{\vec{r}}$.

Treat k and k' which appear in subscripts as the vectors $\vec{k}, \vec{k'}$.

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{\sqrt{\text{Vol}}} \sum_{k\alpha} \left[c_{k\alpha}(\nabla \times \vec{u_{k\alpha}}) + c^*_{k\alpha}(\nabla \times \vec{u^*_{k\alpha}}) \right] \tag{10}$$

We must sidebar to derive

$$\begin{aligned} \nabla \times \vec{u}(\vec{r}) = \nabla \times (e^{i\vec{k}\vec{r}}\epsilon_k); \text{ use } (a \times b)_i = \epsilon_{ijk} a_j b_k \\ (\nabla \times e^{i\vec{k}\vec{r}})_i \rightarrow \epsilon_{ijk}(\frac{d}{dr_j} e^{i\vec{k}\vec{r}}\epsilon^\alpha_k) = i\vec{k} \times e^{i\vec{k}\vec{r}}\epsilon^\alpha_k = i\vec{k} \times \vec{u}(\vec{r}) \end{aligned} \tag{11}$$

Then, the product rule two for cross-products is

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

$$\begin{aligned} \text{Vol} \cdot B^2 &= \sum_{k\alpha} \sum_{k'\beta} (i\vec{k} \times \left[c_{k\alpha}\vec{u_{k\alpha}} - c^*_{k\alpha}\vec{u^*_{k\alpha}} \right])(i\vec{k'} \times \left[c_{k'\beta}\vec{u_{k'\beta}} - c^*_{k'\beta}\vec{u^*_{k'\beta}} \right]) \\ &= \sum_{k\alpha} \sum_{k'\beta} [(-\vec{k} \cdot \vec{k'}) \left[c_{k\alpha}\vec{u_{k\alpha}} - c^*_{k\alpha}\vec{u^*_{k\alpha}} \right] \left[c_{k'\beta}\vec{u_{k'\beta}} - c^*_{k'\beta}\vec{u^*_{k'\beta}} \right] \\ &\quad -(i\vec{k} \cdot \left[c_{k'\beta}\vec{u_{k'\beta}} - c^*_{k'\beta}\vec{u^*_{k'\beta}} \right])(i\vec{k'} \cdot \left[c_{k\alpha}\vec{u_{k\alpha}} - c^*_{k\alpha}\vec{u^*_{k\alpha}} \right])] \end{aligned} \tag{12}$$

We are integrating $\int_{\text{Vol}} B^2 d\vec{r}$, and we can use the identity from problem 3 to simplify this one too: $\int_{\text{Vol}} e^{ikr} = \text{Vol}\delta_{k,0}$.

$$\begin{aligned} \int_{\text{Vol}} e^{i(k+k')r} = \text{Vol}\delta_{k,-k'} \\ \int_{\text{Vol}} e^{i(k-k')r} = \text{Vol}\delta_{k,k'} \\ \int_{\text{Vol}} e^{i(k'-k)r} = \text{Vol}\delta_{k',k} \\ \int_{\text{Vol}} e^{-i(k'+k)r} = \text{Vol}\delta_{k,-k'} \end{aligned} \tag{13}$$

This means that terms such as $\int_{\text{Vol}} d\vec{r}(i\vec{k} \cdot c_{k'\beta}\vec{u}_{k'\beta})(c_{k\alpha}\vec{u_{k\alpha}} \cdot i\vec{k'}) = \int_{\text{Vol}} d\vec{r}(c_{k'\beta} i\vec{k} \cdot e^{i\vec{k'}\cdot\vec{r}}\epsilon^\beta_{k'})(c_{k\alpha}e^{i\vec{k}\cdot\vec{r}}\epsilon^\alpha_k \cdot i\vec{k'})$ using $\int_{\text{Vol}} d\vec{r} e^{i(k+k)} = \text{Vol}\delta_{k,-k'}$ such that we get terms $\vec{k} \cdot \epsilon_k = 0$. As such, all the terms in the $(a \cdot d)(b \cdot c)$ are cancelled out.

$$\begin{aligned} \int_{\text{Vol}} B^2 d\vec{r} = \int_{\text{Vol}} d\vec{r} \frac{1}{\text{Vol}} \sum_{k\alpha} \sum_{k'\beta} (\vec{k} \cdot \vec{k'})[-c_{k\alpha}c_{k'\beta}e^{i(\vec{k}+\vec{k'})\vec{r}}\epsilon^\alpha_k \cdot \epsilon^\beta_{k'} + \\ c_{k\alpha}c^*_{k'\beta}e^{i(\vec{k}-\vec{k'})\vec{r}}\epsilon^\alpha_k \cdot \epsilon^{*\beta}_{k'} + c^*_{k\alpha}c_{k'\beta}e^{-i(\vec{k}-\vec{k'})\vec{r}}\epsilon^{*\alpha}_k \cdot \epsilon^\beta_{k'} - c^*_{k\alpha}c^*_{k'\beta}e^{-i(\vec{k}+\vec{k'})\vec{r}}\epsilon^{*\alpha}_k \cdot \epsilon^{*\beta}_{k'}] \\ = -\sum_{k\alpha\beta} k^2 \left[c_{k\alpha}c_{-k\beta}\epsilon^\alpha_k \cdot \epsilon^\beta_{-k} - c_{k\alpha}c^*_{k\beta}\epsilon^\alpha_k \cdot \epsilon^{*\beta}_k - c^*_{k\alpha}c_{k\beta}\epsilon^{*\alpha}_k \cdot \epsilon^\beta_{k'} + c^*_{k\alpha}c^*_{-k\beta}\epsilon^{*\alpha}_k \cdot \epsilon^{*\beta}_{-k} \right] \end{aligned} \tag{14}$$

Using $\epsilon^{*\alpha}_{-k} = \epsilon^{\alpha}_{k}$ and $\epsilon^{\alpha}_{k} \cdot \epsilon^{*\beta}_{k} = \delta_{\alpha\beta}$, we can simplify further.

$$\begin{aligned}\int_{\text{Vol}} B^2 d\vec{r} &= -\sum_{k\alpha} k^2 \left[c_{k\alpha}c_{-k\alpha} - c_{k\alpha}c^*_{k\alpha} - c^*_{k\alpha}c_{k\alpha} + c^*_{k\alpha}c^*_{-k\alpha}\right] \\ &= \sum_{k\alpha} k^2 \left[c_{k\alpha}c_{-k\alpha} + c_{k\alpha}\overline{c_{k\alpha}} + c.c.\right] \text{ QED}\end{aligned} \tag{15}$$

b. We want to verify coupling modes k, -k are eliminated when we combine $\int_{\text{Vol}} E^2 + B^2 d\vec{r}$. We are given

$$\begin{aligned}\int_{\text{Vol}} E^2 d\vec{r} &= \sum_{m} |k_m||k_{-m}| \left[-c_m c_{-m} - \overline{c_m c_{-m}} + c_m\overline{c_m} + \overline{c_m}c_m\right] \\ \text{and } \int_{\text{Vol}} E^2 d\vec{r} &= \sum_{m} |k_m||k_{-m}| \left[c_m c_{-m} + \overline{c_m c_{-m}} + c_m\overline{c_m} + \overline{c_m}c_m\right] \\ \rightarrow \int_{\text{Vol}} E^2 + B^2 d\vec{r} &= \sum_{m} |k_m||k_{-m}| \left[2c_m\overline{c_m} + 2\overline{c_m}c_m\right]\end{aligned} \tag{16}$$

3 Problem 3

We want to prove that

$$\begin{aligned}\hat{P}_{em} &= \frac{1}{4\pi c}\int_{\text{Vol}} \hat{E} \times \hat{B} d\vec{r} = \sum_{\vec{k}\alpha} \hbar\vec{k}\hat{N}_{\vec{k}\alpha} \text{ with } \hat{N}_{\vec{k}\alpha} = a^*_{\vec{k}\alpha}a_{\vec{k}\alpha} \\ \text{and } \hat{E}(\vec{r}) &= \frac{i}{\sqrt{\text{Vol}}}\sum_{\vec{k}\alpha}\sqrt{h\omega_{\vec{k}}}\left[a_{\vec{k}\alpha}\vec{u}_{\vec{k}\alpha}(\vec{r}) - a^*_{\vec{k}\alpha}\overline{\vec{u}_{\vec{k}\alpha}(\vec{r})}\right] \\ \hat{B}(\vec{r}) &= \frac{i}{\sqrt{\text{Vol}}}\sum_{\vec{k}\alpha}\sqrt{\frac{hc^2}{\omega_{\vec{k}}}}\vec{k} \times \left[a_{\vec{k}\alpha}\vec{u}_{\vec{k}\alpha}(\vec{r}) - a^*_{\vec{k}\alpha}\overline{\vec{u}_{\vec{k}\alpha}(\vec{r})}\right]\end{aligned} \tag{17}$$

$$\hat{E} \times \hat{B} = \frac{hc}{\text{Vol}}\sum_{k\alpha}\sum_{k'\beta}\left[a_{k\alpha}u_{k\alpha} - a^*_{k\alpha}\overline{u}_{k\alpha}\right] \times \left(k' \times \left[a_{k'\beta}u_{k'\beta} - a^*_{k'\beta}\overline{u}_{k'\beta}\right]\right) \tag{18}$$

Recall that $u_{k\alpha} = e^{ikr}\epsilon^{\alpha}_{k}$.

$$\hat{E}\times\hat{B} = \frac{hc}{\text{Vol}}\sum_{k,k',\alpha,\beta}\left[a_{k\alpha}e^{ikr}\epsilon^{\alpha}_{k} - a^*_{k\alpha}e^{-ikr}\epsilon^{*\alpha}_{k}\right]\times\left[a_{k'\beta}e^{ik'r}(\vec{k'} \times \epsilon^{\beta}_{k'}) - a^*_{k'\beta}e^{-ik'r}(\vec{k} \times \epsilon^{*\beta}_{k'})\right] \tag{19}$$

Integrating,

$$\hat{P} = \frac{1}{4\pi c}\int_{\text{Vol}} \frac{hc}{\text{Vol}} \sum_{k,k',\alpha,\beta} \left[a_{k\alpha}e^{ikr}\epsilon_k^{\alpha} - a^*_{k\alpha}e^{-ikr}\epsilon_k^{*\alpha}\right] \times \left[a_{k'\beta}e^{ik'r}(\vec{k'}\times\epsilon_{k'}^{\beta}) - a^*_{k'\beta}e^{-ik'r}(\vec{k'}\times\epsilon_{k'}^{*\beta})\right]$$
$$= \frac{h}{4\pi\text{Vol}}\int_{\text{Vol}} \sum_{k,k',\alpha,\beta} a_{k\alpha}a_{k'\beta}e^{ir(k+k')}(\epsilon_k^{\alpha}\times\vec{k'}\times\epsilon_{k'}^{\beta}) - a_{k\alpha}a^*_{k'\beta}e^{i(k-k')r}(\epsilon_k^{\alpha}\times\vec{k'}\times\epsilon_{k'}^{*\beta})$$
$$-a^*_{k\alpha}a^*_{k'\beta}e^{i(k'-k)r}(\epsilon_k^{*\alpha}\times\vec{k'}\times\epsilon_{k'}^{\beta}) + a^*_{k\alpha}a^*_{k'\alpha}e^{-(k'+k)r}(\epsilon_k^{*\alpha}\times\vec{k'}\times\epsilon_{k'}^{*\beta}) \tag{20}$$

Make use of $\int_{\text{Vol}} e^{ikr} = \text{Vol}\delta_{k,0}$ and $\epsilon_k^{(\alpha)} \times (\vec{k}\times\epsilon_k^{(\alpha')}) = \vec{k}\delta_{aa'}$.

$$\text{Therefore,}\quad \begin{aligned}\int_{\text{Vol}} e^{i(k+k')r} &= \text{Vol}\delta_{k,-k'}\\ \int_{\text{Vol}} e^{i(k-k')r} &= \text{Vol}\delta_{k,k'}\\ \int_{\text{Vol}} e^{i(k'-k)r} &= \text{Vol}\delta_{k',k}\\ \int_{\text{Vol}} e^{-i(k'+k)r} &= \text{Vol}\delta_{k,-k'}\end{aligned} \tag{21}$$

$$\hat{P} = \frac{h}{4\pi}\sum_{k,\alpha,\beta}[a_{k\alpha}a_{-k\beta}(\epsilon_k^{\alpha}\times\vec{k}\times\epsilon_k^{\beta}) - a_{k\alpha}a^*_{k\beta}(\epsilon_k^{\alpha}\times\vec{-k}\times\epsilon_{-k}^{*\beta})$$
$$-a^*_{k\alpha}a_{k\beta}(\epsilon_k^{*\alpha}\times\vec{-k}\times\epsilon_{-k}^{\beta}) + a^*_{-k\alpha}a^*_{k\beta}(\epsilon_k^{*\alpha}\times\vec{k}\times\epsilon_k^{*\beta})] \tag{22}$$

Recalling that $\epsilon_{-k}^{*\alpha} = \epsilon_k^{\alpha}$, we can submit for each $\epsilon_{\pm k}^{\alpha}\times\vec{k}\times\epsilon_{\pm k}^{\beta} = \vec{k}\delta_{\alpha,\beta}$.

$$\hat{P} = \frac{h}{4\pi}\sum_{k\alpha}\vec{k}[a_{k\alpha}a_{-k\alpha} + a_{k\alpha}a^*_{k\alpha} + a^*_{k\alpha}a_{k\alpha} + a^*_{-k\alpha}a^*_{k\alpha}] \tag{23}$$

We sum over positive and negative values of k – meaning that the sum over both $a_{k\alpha}a_{-k\alpha}$ and $a_{-k\alpha}a_{k\alpha}$ is even. Therefore, the multiplication of $\vec{k}$ (an odd function) means that the term $\sum_k \vec{k}a_{k\alpha}a_{-k\alpha} = 0$. This is shown for the number operator with $[a_i, a_j] = [a_k, a_{-k}] = 0$. The same can be said for the c.c.

$$\hat{P} = \frac{h}{4\pi}\sum_{k\alpha}\vec{k}[0 + a_{k\alpha}a^*_{k\alpha} + a^*_{k\alpha}a_{k\alpha} + 0] \tag{24}$$

And using $[a, a^*] = 1 = aa^* - a^*a$

$$\hat{P} = \frac{h}{4\pi}\sum_{k\alpha}\vec{k}[1 + a^*_{k\alpha}a_{k\alpha} + a^*_{k\alpha}a_{k\alpha}] = \frac{1}{2}\hbar\sum_{k\alpha}\vec{k}[1 + 2\hat{N}_{k\alpha}] \tag{25}$$

Similarly to before, the sum over positive/negative k sends $\hbar\vec{k}$ to zero, so we get $\hat{P} = \sum_{k\alpha}\hbar\vec{k}\hat{N}_{k\alpha}$ QED.