1 Problem 1

$$H^0 = \frac{\hat{p}^2}{2m} \otimes (I)_{\text{spin}} + v\hat{p} \otimes \frac{\hat{S}_x}{\hbar/2} \text{ for } v > 0, m > 0 \tag{1}$$

Wavefcns are periodic in $x \in [-L/2, L/2]$

a. Label eigenstates of H^0 as $|p, \pm>$ and energies $E_{p,\pm}$.

$$\frac{\hat{S}_x}{\hbar/2} = \sigma_x \therefore H^0 = \frac{\hat{p}^2}{2m} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + v\hat{p} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

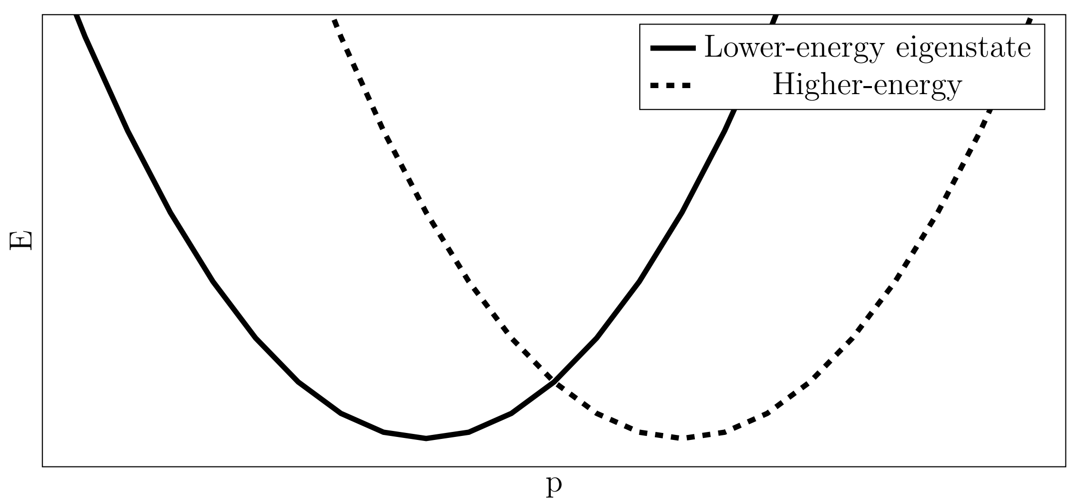
We are looking for the energy eigenstates:

$$\det(\begin{bmatrix} \frac{p^2}{2m} & vp \\ vp & \frac{p^2}{2m} \end{bmatrix} - \lambda I) = 0$$

$$(\frac{p^2}{2m} - \lambda)^2 - (vp)^2 = 0 \rightarrow \lambda = \frac{2\frac{p^2}{2m} \pm \sqrt{4\frac{p^2}{2m} - 4(-4v^2p^2 + \frac{p^2}{2m})}}{2} \tag{2}$$

$$\lambda = \frac{p^2}{2m} \pm vp = E_{p,\pm}$$

Therefore the lowest eigenvalue is $\frac{p^2}{2m} - vp$

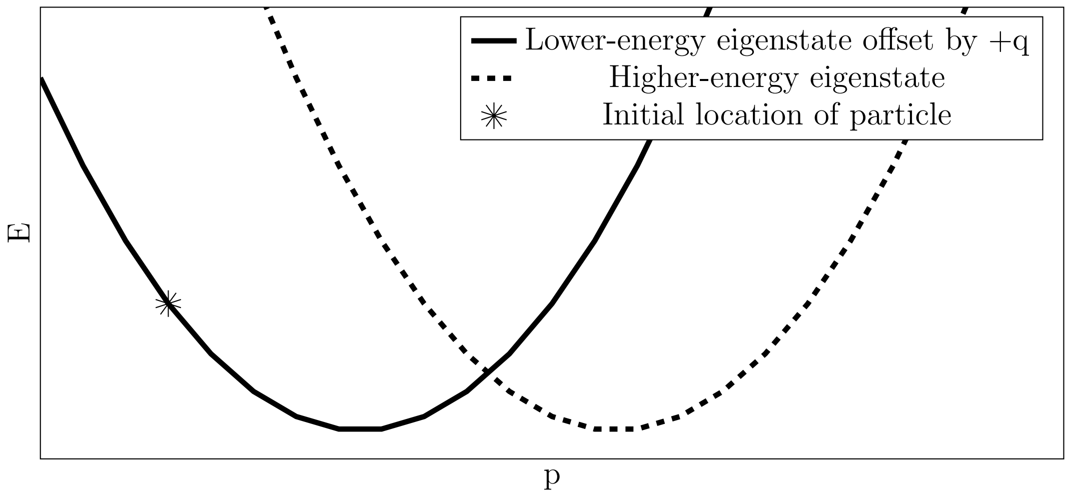


The minima are separated by 2mv. They are offset from the x-axis by $-\frac{1}{2}mv^2$.

$$\text{Ultimately, } E_{p,\pm} = \frac{p^2}{2m} \pm vp = \frac{(p \pm mv)^2}{2m} - \frac{1}{2}mv^2 \tag{3}$$

b. At $t = 0 \rightarrow |p_i, + >$ with $p_i < 0$ and $E_i = \frac{q^2}{2m} - \frac{1}{2}mv^2 > 0$.

$$\begin{gathered} E_i = E_{p,+} \\ \frac{q^2}{2m} - \frac{1}{2}mv^2 = \frac{p^2}{2m} + vp \\ p_i = \frac{-v \pm \sqrt{v^2 - 4\frac{\frac{-q^2}{2m} + \frac{1}{2}mv^2}{2m}}}{2\frac{1}{2m}} \rightarrow p_i = -mv \pm q \end{gathered} \tag{4}$$



There is a separation between the particle location and the minimas of $\frac{1}{2}mv^2$.

c. For time-dep. Schrod. Eqn. with perturbation $V = Ue^{-|\hat{x}|/a} \otimes I_{\text{spin}}$ with particle lifetime τ_i defined by $|C_{p_i}(t)|^2 = |C_{p_i,+}(0)|^2[1 - \frac{1}{\tau_i} + O(t^2)]$.

Use $L >> a : \int_{\frac{-L}{2}}^{\frac{L}{2}} e^{i|x|/a} e^{ikx} dx = \frac{2a}{1+a^2k^2}$.

We use use the Fermi Golden Rule:

$$\frac{1}{\tau_i} = \frac{2\pi}{\hbar} \sum_{f!=i} |V_{fi}|^2 \delta(E_f - E_i) \tag{5}$$

Since the perturbation $V = Ue^{-|\hat{x}|/a} \otimes I_{\text{spin}}$ acts on the spin in the same way as the identity matrix, the spin does not flip. As a result, the only countable branches are those which preserve spin – of which there are two: $|p_i, + >$ and $|p_i + 2q, + >$.

We then perform the sum over branches:

$$\frac{1}{\tau_i} = \frac{2\pi}{\hbar}\Bigg[\sum_{(p,+)!=i} |<p,+|V|i>|^2\delta(E_{p,+}-E_i)+$$
$$\sum_{(p+2q,+)!=i} |<p+2q,+|V|i>|^2\delta(E_{p+2q,+}-E_i)$$
$$= \frac{2\pi}{\hbar}\Bigg[|<p,+|V|i>|^2 \sum_{(p,+)!=i}\delta(E_{p,+}-E_i)+ \tag{6}$$
$$|<p+2q,+|V|i>|^2 \sum_{(p+2q,+)!=i}\delta(E_{p+2q,+}-E_i)$$
$$= \frac{2\pi}{\hbar}\left[|<p,+|V|i>|^2 L\rho(E_i) + |<p+2q,+|V|i>|^2 L\rho(E_i)\right]$$
$$= \frac{2\pi L}{\hbar}\rho(E_i)\left[|<p,+|V|i>|^2 + |<p+2q,+|V|i>|^2\right]$$

Thus two main components to derive are the density of states and the matrix elements V for the two constituent spins.

The density of states is $dN = \frac{\text{Volume}dp}{2\pi\hbar}$ where the volume is L.

$$\rho(E) = \frac{dN}{LdE} = \frac{1}{2\pi\hbar}|\frac{dp}{dE}|$$
$$\text{With the original energy-state}\frac{dE}{dp} = \frac{d}{dp}\frac{p^2}{2m} = \frac{p}{m} \tag{7}$$
$$\rho(E) = \frac{m}{2\pi\hbar p} = \frac{m}{hp}|_{p\to q}$$

Next,

$$<p_i,+|V|p_i,+>= U\int_{\frac{-L}{2}}^{\frac{L}{2}} \frac{e^{i(\frac{p}{\hbar}x)}}{\sqrt{L}} e^{-|\hat{x}|/a}\frac{e^{-i(\frac{p}{\hbar}x)}}{\sqrt{L}}dx = \frac{U}{L}\int e^{-|\hat{x}|/a}dx = \frac{2Ua}{L} \tag{8}$$

$$<p_i+2q,+|V|p_i,+>= \frac{U}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} e^{i(\frac{p+2q}{\hbar}x)}e^{-|\hat{x}|/a}e^{-i(\frac{p}{\hbar}x)}dx = \frac{U}{L}\int e^{-|\hat{x}|/a}e^{\frac{ix}{\hbar}(2q)}dx$$
$$= \frac{U}{L}\frac{2a}{1+a^2(\frac{2q}{\hbar})^2} \tag{9}$$

As such, we can solve for

$$\sum_{f!=i}|V_{fi}|^2 = (\frac{U}{L})^2\left[|2a|^2 + |\frac{2a}{1+(\frac{2qa}{\hbar})^2}|^2\right] \tag{10}$$

Putting this all together, we get

$$\begin{aligned}\frac{1}{\tau_i} &= \frac{2\pi L}{\hbar}\frac{m}{2\pi\hbar q}(\frac{U}{L})^2\left[|2a|^2 + |\frac{2a}{1+(\frac{2qa}{\hbar})^2}|^2\right] \\ &= \frac{4U^2ma^2}{L\hbar^2 q}\left[1+\frac{1}{|1+(\frac{2qa}{\hbar})^2|^2}\right]\end{aligned} \tag{11}$$

d. Denoting the units of a quantity with [qty].

- $[m]$ = mass, [L] = length; therefore [mv] = mass$\cdot\frac{\text{length}}{\text{seconds}(s)}$
- $q \propto p \propto mv$ therefore [q]=[mv]
- $V \propto$ energy $\rightarrow [U]$ = energy
- $[\hbar]$ = energy $\cdot$ (s)
- Lastly, [a]=length to make $e^{|x|/a}$ scale with the perturbation

As such, take the constant outside the brackets of the above eqn.:

$$[\frac{4U^2ma^2}{L^2\hbar^2 q}] = \frac{(\text{energy})^2\text{mass}(\text{length})^2}{(\text{length})(\text{energy})^2 s^2\frac{(\text{mass})(\text{length})}{s}} = \frac{1}{s} \tag{12}$$

Inside the brackets,

$$[(\frac{2qa}{\hbar})^2] = \frac{\frac{(\text{mass})(\text{length})}{s}(\text{length})}{\text{energy}\cdot s} = \frac{\text{energy}}{\text{energy}} \rightarrow \text{unitless} \tag{13}$$

Therefore, the whole eqn. is of units 1/s.

e. Change perturbation $V \rightarrow V' = Ue^{-|x|/a}\otimes\frac{\hat{S}_z}{\hbar/2}$ for new lifetime $\frac{1}{\tau_i'}$.

Spin will now flip because the perturbation is in $\hat{S}_x$; contrary to our initial states in the $\hat{z}$ basis. As with the figure in **part b.**, the parabolas are separated by a gap (2mv + q). As such, moving from any state in the spin + state will require a change

2mv or 2mv+2q. Therefore, our branches are $|p_i+2mv,->$ and $|p_i+2mv+2q,->$.

$$< p_i + 2mv, +|V|p_i, + >= \frac{U}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} e^{i(\frac{p+2mv}{\hbar}x)}e^{-|\hat{x}|/a}e^{-i(\frac{p}{\hbar}x)}dx = \frac{U}{L}\int e^{-|\hat{x}|/a}e^{\frac{ix}{\hbar}(2mv)}dx$$
$$= \frac{U}{L}\frac{2a}{1+a^2(\frac{2mv}{\hbar})^2} \tag{14}$$

$$< p_i + 2mv + 2q, +|V|p_i, + >= \frac{U}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} e^{i(\frac{p+2mv+2q}{\hbar}x)}e^{-|\hat{x}|/a}e^{-i(\frac{p}{\hbar}x)}dx = \frac{U}{L}\int e^{-|\hat{x}|/a}e^{\frac{ix}{\hbar}(2mv+2q)}dx$$
$$= \frac{U}{L}\frac{2a}{1+a^2(\frac{2mv+2q}{\hbar})^2} \tag{15}$$

Where assembling this into the Fermi Golden Rule as before:

$$\frac{1}{\tau_i'} = \frac{2\pi L}{\hbar}\rho(E_f)\sum_{f!=i}|V_{fi}|^2 = \frac{4a^2U^2m}{L\hbar^2 q}\left[|\frac{1}{1+a^2(\frac{2mv}{\hbar})^2}|^2 + |\frac{1}{1+a^2(\frac{2mv+2q}{\hbar})^2}|^2\right] \tag{16}$$

f. Let's compare the two results:

$$\frac{1}{\tau_i} == \frac{1}{\tau_i'}$$
$$\frac{4U^2ma^2}{L\hbar^2 q}\left[1+\frac{1}{|1+(\frac{2qa}{\hbar})^2|^2}\right] == \frac{4a^2U^2m}{L\hbar^2 q}\left[|\frac{1}{1+a^2(\frac{2mv}{\hbar})^2}|^2 + |\frac{1}{1+a^2(\frac{2mv+2q}{\hbar})^2}|^2\right]$$
$$\left[1+|\frac{1}{1+a^2(\frac{2q}{\hbar})^2}|^2\right] == \left[|\frac{1}{1+a^2(\frac{2mv}{\hbar})^2}|^2 + |\frac{1}{1+a^2(\frac{2mv+2q}{\hbar})^2}|^2\right] \tag{17}$$

Since the right side terms have more in the denominator (i.e. 2mv+2q on the RHS compared to just 2mv on the LHS) and all the fraction terms are < 1:

$$\frac{1}{1+a^2(\frac{2q}{\hbar})^2} > \frac{1}{1+a^2(\frac{2mv+2q}{\hbar})^2}$$
$$\text{and } 1 > \frac{1}{1+a^2(\frac{2mv}{\hbar})^2} \tag{18}$$

Altogether then, $\frac{1}{\tau_i} > \frac{1}{\tau_i'}$.

2 Problem 2

Identical atoms, each with an e^- in $|A>$ or $|B>$, corresponding to $(H_{\text{atom}} - E_A)|A>= (H_{\text{atom}} - E_B)|B>= 0, E_A > E_B$. They exchange energy via a linearly polarized photon mode $A \leftrightarrow B + \gamma_m$ where mode m is specified by a wavevector k and polarization vector ϵ_k^α orthogonal to k. The forward process indicates emission of energy $E_A - E_B = \hbar\omega_m$ with $\omega_m = c|k|$; the backward process indicates absorption.

$$\begin{gathered} H = H^0 + V \\ H^0 = H_{\text{atom}} + \hbar\omega_m a_m^* a_m \\ V = \frac{|e|}{m}\sqrt{\frac{h}{\text{Vol}\omega_m}}[a_m e^{ik\hat{r}} + a_m^* e^{-ik\hat{r}}]\epsilon_k^\alpha \cdot \hat{p} \end{gathered} \tag{19}$$

a. The exchange of energy is mediated by the photon field, which acts as a heat reservoir for the atoms (free & reversible interaction). When the system is in thermal equilibrium, we can approximate the probability distribution of the atoms' energies with e^{-E/k_BT}. Therefore, Boltzmann's factor is

$$\begin{gathered} \frac{N(B)}{N(A)} = e^{-E_B/(k_BT)}e^{E_A/(k_BT)} \text{ where } E_A - E_B = \hbar\omega_m \\ \therefore \frac{N(B)}{N(A)} = e^{\frac{\hbar\omega_m}{k_BT}} \end{gathered} \tag{20}$$

b. The emission process $|A> \otimes |n_m> \to |B> \otimes |n_m + 1>$ with $|n_m>$ denoting Fock space and n_m the number of photons in mode m.

Recall for the annihilation operator, $a|n>= \sqrt{n}|n-1$ & $a^*|n>= \sqrt{n+1}|n+1>$.

Using the shorthand $|A> \otimes |n_m>= |A, n_m>$:

$$\begin{aligned} < B, n_m + 1|V|A, n_m >&= \frac{|e|}{m}\sqrt{\frac{h}{\text{Vol}\omega_m}} < B, n_m + 1|a_m e^{ik\hat{r}}\epsilon_k^\alpha \cdot \hat{p} + a_m^* e^{-ik\hat{r}}\epsilon_k^\alpha \cdot \hat{p}|A, n_m > \\ &= \frac{|e|}{m}\sqrt{\frac{h}{\text{Vol}\omega_m}}[< B|e^{ik\hat{r}}\epsilon_k^\alpha \cdot \hat{p}|A > \cdot < n_m + 1|a_m|n_m > \\ &\quad + < B|e^{-ik\hat{r}}\epsilon_k^\alpha \cdot \hat{p}|A > \cdot < n_m + 1|a_m^*|n_m >] \\ &= \frac{|e|}{m}\sqrt{\frac{h}{\text{Vol}\omega_m}}[< B|e^{ik\hat{r}}\epsilon_k^\alpha \cdot \hat{p}|A > \cdot\sqrt{n_m} < n_m + 1|n_m - 1 > \\ &\quad + < B|e^{-ik\hat{r}}\epsilon_k^\alpha \cdot \hat{p}|A > \cdot\sqrt{n_m + 1} < n_m + 1|n_m + 1 >] \end{aligned} \tag{21}$$

Now only the element $< n_m + 1|n_m + 1 >= 1$; the other term goes to zero.

$$< B, n_m + 1|V|A, n_m >= \frac{|e|}{m}\sqrt{\frac{h}{\text{Vol}\omega_m}}\sqrt{n_m+1} < B|e^{-ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha|A > \tag{22}$$

c. Now for the absorption $|B > \otimes|n_m > \rightarrow |A > \otimes|n_m - 1 >$:

$$\begin{aligned} < A, n_m - 1|V|B, n_m >= \frac{|e|}{m}\sqrt{\frac{h}{\text{Vol}\omega_m}}[< A|e^{ik\hat{r}}\epsilon_k^\alpha\cdot\hat{p}|B > \cdot < n_m - 1|a_m|n_m > \\ + < A|e^{-ik\hat{r}}\epsilon_k^\alpha\cdot\hat{p}|B > \cdot < n_m - 1|a_m^*|n_m >] \\ = \frac{|e|}{m}\sqrt{\frac{h}{\text{Vol}\omega_m}}[< A|e^{ik\hat{r}}\epsilon_k^\alpha\cdot\hat{p}|B > \cdot\sqrt{n_m} < n_m - 1|n_m - 1 > \\ + < A|e^{-ik\hat{r}}\epsilon_k^\alpha\cdot\hat{p}|B > \cdot\sqrt{n_m+1} < n_m - 1|n_m + 1 >] \end{aligned} \tag{23}$$

Therefore,

$$< A, n_m - 1|V|B, n_m >= \frac{|e|}{m}\sqrt{\frac{h}{\text{Vol}\omega_m}}\sqrt{n_m} < A|e^{ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha|B > \tag{24}$$

d. For emission: $\frac{dP_{(B,n_m+1)\leftarrow(A,n_m)}}{dt} = \frac{2\pi}{\hbar}\text{Vol}\frac{\rho(\hbar\omega_m)}{4\pi}| < B, n_m + 1|V|A, n_m > |^2$;
For absorption: $\frac{dP_{(A,n_m-1)\leftarrow(B,n_m)}}{dt} = \frac{2\pi}{\hbar}\text{Vol}\frac{\rho(\hbar\omega_m)}{4\pi}| < A, n_m - 1|V|B, n_m > |^2$
For thermal equilibrium, N(A) is time-independent, therefore

$$\frac{dP_{(B,n_m+1)\leftarrow(A,n_m)}}{dt}N(A) = \frac{dP_{(A,n_m-1)\leftarrow(B,n_m)}}{dt}N(B)$$

Plugging in **a.-c.** (their constants cancel!):

$$\begin{aligned} |\sqrt{n_m+1}| < B|e^{-ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha|A > |^2 N(A) = |\sqrt{n_m} < A|e^{ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha|B > |^2 N(B) \\ |\frac{< B|e^{-ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha|A >}{< A|e^{ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha|B >}|^2 = \frac{n_m}{n_m+1}\frac{N(B)}{N(A)} = \frac{n_m}{n_m+1}e^{\frac{\hbar\omega_m}{k_B T}} \end{aligned} \tag{25}$$

If we take the complex conjugate of the numerator, we get $< A|(e^{ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha)^*|B >$.

$$\begin{gathered}(e^{ik\hat{r}}\hat{p}\cdot\epsilon)^* = \epsilon\cdot\hat{p}(e^{-ik\hat{r}}) \text{ recalling } \hat{p} = i\hbar\frac{d}{d\hat{r}} \\ = i\hbar\left[e^{-ik\hat{r}}\epsilon\frac{d}{d\hat{r}} + \epsilon\frac{d}{d\hat{r}}\{e^{-ik\hat{r}}\}\right] \text{ where } \{\} \text{ denotes derivative.} \\ \text{Focusing in on } \epsilon\frac{d}{d\hat{r}}e^{-ik\hat{r}} = \sum_{\alpha,\beta}\epsilon_\alpha\cdot\frac{d}{d\hat{r}_\alpha}\{e^{ik_\beta\hat{r}_\beta}\} = \sum_{\alpha,\beta}\epsilon_\alpha\cdot\frac{d}{d\hat{r}_\alpha}\{ik_\beta\cdot\hat{r}_\beta\}e^{ik_\beta\hat{r}_\beta} \\ \text{Which, using } \frac{d}{d\hat{r}_\alpha r_\beta = \delta_{\alpha\beta}}, \epsilon\frac{d}{d\hat{r}}e^{-ik\hat{r}} = \sum_{\alpha,\beta}\epsilon_\alpha\cdot\{ik_\beta\}\delta_{\alpha\beta}e^{ik_\beta\hat{r}_\beta} \\ \text{Wherein the resulting } \epsilon_\alpha\cdot k_\alpha = 0 \text{ due to orthogonality,} \\ \text{therefore, only the first term remains and} \\ (e^{ik\hat{r}}\hat{p}\cdot\epsilon)^* = e^{-ik\hat{r}}(\epsilon\cdot\hat{p})\end{gathered} \tag{26}$$

Using the above result, $|\frac{<B|e^{-ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha|A>}{<A|e^{ik\hat{r}}\hat{p}\cdot\epsilon_k^\alpha|B>}|^2 = 1$ and

$$\begin{gathered}\frac{n_m}{n_m+1}e^{\frac{\hbar\omega_m}{k_BT}} = 1 \\ n_m = \frac{1}{e^{\frac{\hbar}{k_BT}\omega_m}-1}\end{gathered} \tag{27}$$

Which is Planck's radiation law wrt ω_m. This is a big deal because it is an alternative derivation from the Quantum Field Theory perspective of Black Body radiation. This means two things: First, Planck's law was empirically fitted; this proves Planck's result. Second, we use a value $\hbar\omega = E_A - E_B$ in which light is quantized, an intuition which Planck could not confirm until Einstein developed this approach years later.